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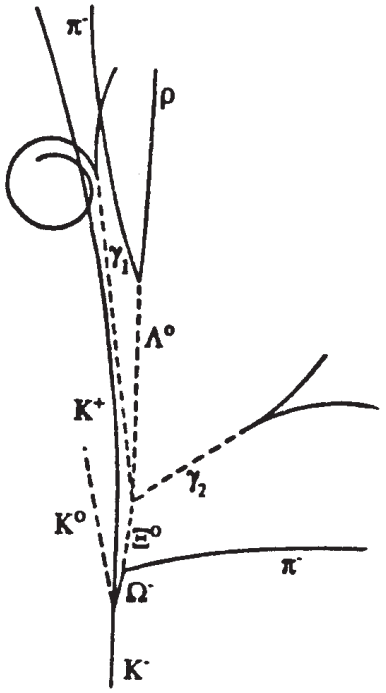

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HELICITY-0 SPINNING PARTICLES[1]

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Abstract

We show that a self-consistent classical theory of the spin, based on a very general Lagrangian extending the Newtonian dynamics, does predict the special case of helicity-0 particles, which at the same time are endowed with nonzero spin and zero intrinsic angular momentum.

Keywords: spin and helicity, special relativity, semiclassical theories, longitudinal polarizations, axial particles, Lorentz invariance, Dirac theory and Zitterbewegung

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1 Introduction

Besides the ordinary cases of spinless scalars and spinnning spinors and vectors, it exists another exotic class of elementary particles: the helicity-0 particles (HZPs), which are spinning, but with zero helicity and vanishing conserved component of the spin vector. HZPs have occasionally been associated to longitudinal excitations of vector fields found in string theory and in some dark-matter and dark-energy cosmological theories (sometimes recognized as "ghost" fields or "aether"-like fields). In particular, both longitudinal and trasversal components of vector fields with a nonzero covariant divergence have been considered in the context of general relativity for describing the expansion of the Universe in the background of the de Sitter cosmological metric. The energy-momentum tensor of the most simple zero-mass longitudinal vector fields enters the Einstein equations as a kind of cosmological constant: therefore the longitudinal waves could really contribute to the dark energy arising [1]. Another known approach to describe longitudinal spin-1 waves is the Stückelberg theory [2], which recovers the longitudinal degree of freedom lost in the Proca theory for a massive vector field (due to the trasversal Lorentz gauge), in such a way restoring the U(1) gauge invariance. Furthermore, in [3] it is shown that a massive vector particle can exist only in the helicity-0 state if it is composed of a fermion-antifermion pair (the mass being generated by the actractive interaction between the two massless particles). In that paper it is also proved that a massive helicity-0 vector particle should behave as a pseudoscalar and that could be identified experimentally by detecting an asymmetry in its decay products, since the longitudinal polarization can lead to forward-backward asymmetries.[3]

Let us also notice that the Lorentz symmetry of vacuum quantum fields, usually satisfied by Klein–Gordon spin-zero fields (as e.g. the ordinary Higgs boson), *is not broken by HZPs*, because for such particles the quantum vacuum expectation value of the spin vector (equal, as is known, to the helicity) vanishes in any frame. Therefore the helicity-0 field could eventually play

[3]Furthermore, unlike a charged particle with spin 1/2 or spin 1, a charged helicity-0 particle will have zero magnetic moment and its direction of polarization will not be altered by a magnetic field.

the rôle of a proper quantum vacuum field.

In the next section we shall analyze the arising of a nonvanishing spin vector due to a boost applied to the center-of-mass frame (CMF), a sort of "extrinsic" spin analogous to the orbital angular momentum. Then we shall see that, differently from scalar spinless particles, at a classical level HZPs own internal degrees of freedom so that they can be considered as extended-like particles. Actually, we shall prove that they are special solutions of a very general classical model for free spinning particles corresponding to degenerate oscillatory motions in the CMF.

2 Extrinsic spin vector

For a spinning particle with 4-momentum $p^\mu = (p^0; \boldsymbol{p})$, total angular momentun $J^{\mu\nu}$, spin tensor $S^{\mu\nu}$, and spin 3-vector $\boldsymbol{s}$ the (conserved) Pauli–Lubanski 4-vector is defined as

$$W^\mu \equiv \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} p_\sigma = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} S_{\nu\rho} p_\sigma = (\boldsymbol{s}\cdot\boldsymbol{p}\,;\; p^0\boldsymbol{s} - \boldsymbol{p}\times\boldsymbol{k}) \tag{1}$$

where $\boldsymbol{k}$ is the Lorentz-boost generator (in the particle "inner" spin space)

$$S_{\mu\nu} \equiv \begin{pmatrix} 0 & k^1 & k^2 & k^3 \\ -k^1 & 0 & -s^3 & s^2 \\ -k^2 & s^3 & 0 & -s^1 \\ -k^3 & -s^2 & s^1 & 0 \end{pmatrix} \tag{2}$$

The square of W^μ is a Casimir-invariant of the Poincaré group and is equal[4] to $-m^2\boldsymbol{s}_\star^2$, indicating by $\boldsymbol{s}_\star$ the spin vector in the CMF where $p^\mu = (m;0,0,0)$ and $W^\mu = (0; m\boldsymbol{s}_\star)$. If in the CMF the spin vector vanishes, $\boldsymbol{s}_\star = 0$, there it vanishes also the Pauli–Lubanski 4-vector, $W^\mu = (0;0,0,0)$. In that case W^μ *is zero in any reference frame*:

$$\boldsymbol{s}\cdot\boldsymbol{p} = p^0\boldsymbol{s} - \boldsymbol{p}\times\boldsymbol{k} = 0 \tag{3}$$

[4]We adopt the signature $+---$ for the metric; hereafter $c = \hbar = 1$.

It follows that in a generic reference frame we shall have

$$\boldsymbol{s} = \boldsymbol{w} \times \boldsymbol{k} \tag{4}$$

where $\boldsymbol{w} \equiv \boldsymbol{p}/p^0$ is the center-of-mass speed[5]. Actually the spin $\boldsymbol{w} \times \boldsymbol{k}$ results always orthogonal to $\boldsymbol{k}$[6] and to $\boldsymbol{w}$, and then also to the momentum, as expected from the vanishing of the helicity. We notice that even for $\boldsymbol{s}_\star = 0$, if $\boldsymbol{k}_\star \neq 0$, the spin vector is not necessarily vanishing in a reference frame other than the CMF: namely, in principle *we can have particles with zero intrinsic angular momentum, i.e. spin zero in the CMF, but with nonzero spin in other reference frames.* This case has not been sufficiently studied in the literature. Notice also that for such particles the momentum is orthogonal to the spin and the helicity is zero, $\boldsymbol{s} \cdot \boldsymbol{p} = 0$, *in any frame.* Let us remark that, only in this case, that is when also the spatial part of the Pauli-Lubanski 4-vector vanishes, *the helicity is a relativistic invariant quantity.* Hence, in this special case the (zero) *helicity results to be an intrinsic property*, just as spin modulus or electrical charge which classify a given particle and are observer-independent.

Elementary particles with vanishing intrinsic angular momentum and helicity, then obeying Eq. (3), belong to two different classes:

a) particles with spin tensor $S^{\mu\nu}$ identically zero (and then $\boldsymbol{s} = \boldsymbol{k} = 0$)

[5]Of course, Eq, (4) can also be obtained from the Lorentz-transformated $S^{\mu\nu}$, applying a boost $-\boldsymbol{w}$ to the CMF (as a consequence the particle's center-of-mass, at rest in the CMF, acquires a velocity $\boldsymbol{w}$ in the new frame) and taking into account that $\boldsymbol{s}_\star = 0$

$$\boldsymbol{k} = \gamma \boldsymbol{k}_\star$$

$$\boldsymbol{s} = \gamma \boldsymbol{w} \times \boldsymbol{k}_\star = \boldsymbol{w} \times \boldsymbol{k}$$

where as usual $\gamma = 1/\sqrt{1 - \boldsymbol{w}^2}$ indicates the Lorentz factor.

[6]From the orthogonality between $\boldsymbol{s}$ and $\boldsymbol{k}$ it also follows

$$e^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} \equiv \widetilde{S} S = \boldsymbol{s} \cdot \boldsymbol{k} = 0$$

For the other Lorentz invariant quantity built up with the spin tensor we find

$$S^{\mu\nu} S_{\mu\nu} = \boldsymbol{s}^2 - \boldsymbol{k}^2 = -\boldsymbol{k}_\star^2$$

in all frames. This is the case of ordinary *scalar spinless* particles described by the Klein–Gordon equation

b) particles with nonvanishing spin tensor ($S^{\mu\nu}$, $\boldsymbol{s}$, $\boldsymbol{k} \neq 0$) in a generic reference frame (in the CMF only the polar part of $S^{\mu\nu}$, i.e. $\boldsymbol{k}$, is different from zero). These helicity-0 particles are non-scalars also if, just like Klein–Gordon scalars, are endowed with only one degree of freedom.

In a sense we could say that the particles described in b) seem to own, besides the usual orbital angular momentum $\boldsymbol{l} \equiv \boldsymbol{x} \times \boldsymbol{p}$, also a kind of "extrinsic" spin vector, which arises only in the presence of an "external" motion of the center-of-mass, and disappears in the CMF, just like the orbital momentum does.

3 Classical spin

Quantum mechanics of the particles which constitute the ordinary matter (spin–$\frac{1}{2}$ particles as quark or electrons) presents a phenomenon known as *Zitterbewegung* (ZBW), described for the first time by Schrödinger in the 1930s [4, 5, 6, 7, 8, 9, 10]: namely, the independence between velocity and momentum (even without external forces)

$$v \neq \frac{p}{m} \tag{5}$$

Actually in Dirac theory the velocity and momentum operators are not proportional

$$\widehat{\boldsymbol{v}} = \boldsymbol{\alpha} \neq \widehat{\boldsymbol{p}} = -i\boldsymbol{\nabla} \tag{6}$$

Furthermore $\widehat{\boldsymbol{v}}$, differently from $\widehat{\boldsymbol{p}}$, does not commute with the Dirac Hamiltonian $\widehat{\mathcal{H}} = \boldsymbol{\alpha} \cdot \widehat{\boldsymbol{p}} + m\beta$ so that, while $\boldsymbol{p}$ is a constant quantity, $\boldsymbol{v}$ is not. Therefore *the dynamical nature of quantum mechanics is intrinsically non-Newtonian*, as it appears more evident after time-derivating side by side the previous equation

$$\boldsymbol{v} \neq \frac{\boldsymbol{p}}{m} \implies \boldsymbol{a} \neq \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\boldsymbol{F}}{m} \tag{7}$$

Thus, for microsystems in general *the Newton's Law and (in the absence of external forces) the Galileo's Principle of Inertia do not apply anymore*, and the free motion is not in general uniform rectilinear. The ZBW is fully depicted through the celebrated Gordon deccomposition [11] of the Dirac probability current (here $\hbar = 1$):

$$j^{\mu} = \overline{\psi}\gamma^{\mu}\psi = \frac{1}{2m}\left[\overline{\psi}(\widehat{p}^{\mu}\psi) - (\widehat{p}^{\mu}\overline{\psi})\psi\right] + \frac{1}{m}\partial_{\nu}\left(\overline{\psi}\widehat{S}^{\mu\nu}\psi\right) \tag{8}$$

($\overline{\psi} \equiv \psi^{\dagger}\gamma^{0}$, $\widehat{p}^{\mu} \equiv i\partial^{\mu}$, e $\widehat{S}^{\mu\nu} \equiv i(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})/4$ represents the spin tensor operator). The first term in the r.h.s. is associated with the "external", translational motion of the center-of-mass; whilst, the non-Newtonian term in the r.h.s. is related to the spin, and describes the "internal", ZBW rotational motion. Actually, with the separation between the global motion of the charge and the mean motion of the center-of-mass, a spinning particle appears to be as "extended-like", occupying in the CMF a region with dimensions of the order of the Compton radius $1/2m$. This extended-like structure of non-Newtonian particles has an impressive counterpart in the famous "Darwin term" which appears in the non-relativistic approximation of the Dirac equation in the presence of an external electromagnetic field, found by C.G. Darwin [12] already in 1928. Indeed, because of the ZBW the electron undergoes extremely rapid fluctuations on scales just of the order of the Compton wavelength, causing, for example, the electrons moving inside an atom to experience a smeared nuclear Coulomb potential.

Analogous Gordon-like decompositions of the conserved 4-currents can be found also for spin-1 bosons and for spin-$\frac{3}{2}$ fermions in the Proca and Rarita-Schwinger theories, respectively.

For what seen so far, we presume that any classical (both relativistic and not) theory of spinning particles must predict the ZBW motion, as a signature of the presence of spin freedom degrees. In a paper of ours[13] it was proposed a *classical* particle theory in which spin and ZBW arise quite naturally. For the above considerations we called that theory *Non-Newtonian Mechanics* (NNM). The classical motion of spinning particles was therein described without recourse to particular models or special formalisms, and without employing Clifford algebras, or classical spinors (appearing in all supersymmetric-like classical models, as the Barut-Zanghi one[8]), but sim-

ply by generalizing the usual spinless theory. It was only assumed the invariance with respect to the Poincaré group, and only requiring the conservation of the linear and angular momenta we derived the ZBW and the other kinematical properties and motion constraints. Standard Newtonian mechanics[7] is re-obtained as a particular case of that theory: namely for spinless systems with no ZBW.

As abovesaid, in the absence of external forces, the Lagrangian shall be invariant with respect to the Poincaré group: thus it must be a function of only scalar quantities which do not depend on particle position but instead on 4-velocity and, in case, also on its derivatives. We can built up, also in the presence of external fields, a covariant Lagrangian formulation of the NNM through a straightforward generalization of the Newtonian Lagrangian $\mathcal{L}^{(0)} = \frac{1}{2}mv^2 - U$ ($v^2 \equiv v_\mu v^\mu$) to Lagrangians containing proper[8] time-derivatives of the velocity up to the n-th order:

$$\mathcal{L}^{(n)}(\tau; x, v, \dot{v}, \ddot{v}, \ldots) \equiv \frac{1}{2}mv^2 + \frac{1}{2}k_1\dot{v}^2 + \frac{1}{2}k_2\ddot{v}^2 + \cdots - U \equiv \sum_{i=0}^{n} \frac{1}{2}k_i v^{(i)^2} - U \tag{9}$$

where $U(x)$ is a (τ-independent) scalar potential due to external forces, the k_i are constant scalar coefficients endowed with alternate signs, $k_0 = m$, and $v^{(i)} \equiv \mathrm{d}^i v/\mathrm{d}\tau^i$[9]. The coefficients k_i —which may be chosen equal to zero for i larger than a given integer, see below— might be functions of the self-interaction of the particle and of its mass and charge (cf. the infinite-terms equation of the self-radiating classical electron or the "chronon" theory of the electron[14], k_i can be related to the underlying string structure (or membrane or n-brane structure) of a spinning particle. The consequent Euler-Lagrange equation of motion is a constant-coefficients n-th order differential equation,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \dot{\mathcal{L}}}{\partial \dot{x}} - \frac{\partial \ddot{\mathcal{L}}}{\partial \ddot{x}} + \frac{\partial \dddot{\mathcal{L}}}{\partial \dddot{x}} - \cdots \tag{10}$$

[7]Hereafter we shall call "Newtonian" the ordinary (relativistic or not) theory, where momentum and velocity are parallel vectors.

[8]Quantity τ is the proper time, i.e. the CMF time.

[9]Alternate signs are requested for the coefficients of the terms appearing in the Lagrangian if we want only stationary solutions and finite oscillatory motions.

which appears as a generalization of the Newton's Law $F = ma$, in which non-Newtonian ZBW terms appear:

$$-\frac{\partial U}{\partial x_\mu} = m\,a^\mu - k_1\,\ddot{a}^\mu + k_2\,\dddot{a}^\mu - \cdots \equiv \sum_{i=0}^{n} (-1)^{\mathrm{i}} k_i\, a^{(2\mathrm{i})\mu} \tag{11}$$

The canonical momentum $\dfrac{\partial \mathcal{L}}{\partial \dot{x}_\mu} - \dfrac{\partial \dot{\mathcal{L}}}{\partial \ddot{x}_\mu} + \dfrac{\partial \ddot{\mathcal{L}}}{\partial \dddot{x}_\mu} - \cdots$ conjugate to x^μ writes

$$p^\mu = m\,v^\mu - k_1\,\ddot{v}^\mu + k_2\,\dddot{v}^\mu - \cdots \equiv \sum_{i=0}^{n} (-1)^{\mathrm{i}}\, k_i\, v^{(2\mathrm{i})\mu}\,, \tag{12}$$

from which we get *the ZBW equation* for $\mathcal{L}^{(n)}$:

$$v^\mu = \frac{p^\mu}{m} + \frac{k_1}{m}\,\ddot{v}^\mu - \frac{k_2}{m}\,\dddot{v}^\mu - \cdots = \frac{p^\mu}{m} - \sum_{i=1}^{n} (-1)^{\mathrm{i}}\,\frac{k_i}{m}\,v^{(2\mathrm{i})\mu}\,. \tag{13}$$

Let us remark that the above ZBW equation has been derived in [13] in a more general form [of which Eq. (13) is a particular case] starting from conservation of the linear and angular momenta:

$$v^\mu = \frac{p^\mu}{m} - \frac{\dot{S}^{\mu\nu} p_\nu}{m^2} \tag{14}$$

In the previously mentioned paper, we also showed that, with the assumption $k_1 = -1/4m$, the first-order NNM, namely $\mathrm{NNM}^{(1)}$, is the very classical analogue of the Dirac theory. As a matter fo fact, the non-newtonian classical equations of motion turn out to be identical to the ones holding for the corresponding quantum operators: we can say that the canonical representation of the $\mathrm{NNM}^{(1)}$ represents a suitable classical counterpart of the Dirac algebra. The Dirac ZBW motion which yields the expected value $s = \frac{1}{2}$ is lightlike (of course in $\mathrm{NNM}^{(1)}$ they are possible also other motions, also not uniform). Actually, the "inner" (i.e. in the CMF) motion of the pointlike charge in a plane normal to the spin vector is uniform circular with constant speed c, even if the ZBW cycle average speed w (the

"external" center-of-mass speed) results to be, as expected, always subluminal[10]. Henceforth let us consider free particles, with absence of external forces $U = 0$. Through the Nöther Theorem, by satisfying the symmetry under 4-rotations, the classical spin can be univocally defined employing only classical kinematical quantities. Actually, requiring Lorentz symmetry for any $\mathcal{L}^{(n)}(\tau; x, v, \dot{v}, \ddot{v}, \ldots)$, we obtain [13] the conservation of the total angular momentum $J^{\mu\nu}$. Hence, we shall have for the ordinary Newtonian case (the only spinless) $\mathcal{L}^{(0)}$

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{15}$$

$$\boldsymbol{s} = 0 \tag{16}$$

The spin arises from the first order forward: for $\mathcal{L}^{(1)}$ we get

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + k_1 \left(v^\mu a^\nu - v^\nu a^\mu\right) \tag{17}$$

hence

$$S^{\mu\nu} = k_1 \left(v^\mu a^\nu - v^\nu a^\mu\right) \qquad \boldsymbol{s} = k_1 \left(\boldsymbol{v} \times \boldsymbol{a}\right) \tag{18}$$

The spin vectors for $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(3)}$ are, respectively

$$\boldsymbol{s} = k_1 \left(\boldsymbol{v} \times \boldsymbol{a}\right) + k_2 \left(\boldsymbol{a} \times \dot{\boldsymbol{a}} - \boldsymbol{v} \times \ddot{\boldsymbol{a}}\right) \tag{19}$$

$$\boldsymbol{s} = k_1(\boldsymbol{v} \times \boldsymbol{a}) + k_2(\boldsymbol{a} \times \dot{\boldsymbol{a}} - \boldsymbol{v} \times \ddot{\boldsymbol{a}}) + k_3(\dot{\boldsymbol{a}} \times \ddot{\boldsymbol{a}} - \boldsymbol{a} \times \dddot{\boldsymbol{a}} + \boldsymbol{v} \times \ddddot{\boldsymbol{a}}) \tag{20}$$

For a generic n (≥ 1) spin tensor and spin vector write, respectively:

$$S_{\mu\nu} = \sum_{i=1}^{n} k_i \sum_{l=0}^{i-1} (-1)^{i-l-1} \left(v_\mu^{(l)} v_\nu^{(2i-l-1)} - v_\nu^{(l)} v_\mu^{(2i-l-1)}\right) \tag{21}$$

$$\boldsymbol{s} = \sum_{i=1}^{n} k_i \sum_{l=0}^{i-1} (-1)^{i-l-1} \boldsymbol{v}^{(l)} \times \boldsymbol{v}^{(2i-l-1)} \tag{22}$$

[10]Note that for Dirac ZBW uniform motions we have, just as it occurs for the electromagnetic tensor of freely propagating waves in vacuum, $S_{\mu\nu} S^{\mu\nu} = \widetilde{S}^{\mu\nu} S_{\mu\nu} = 0$.

4 Zitterbewegung oscillations with vanishing intrinsic angular momentum and helicity

A very interesting case of $\mathrm{NNM}^{(n)}$ is the one that, for any n, entails only *harmonic motions endowed with only one frequency* ω_0 with multiplicity n:

$$\omega_i \;=\; \omega_0 \qquad\qquad n = 1 \ldots n \tag{23}$$

As is shown in [13], the periodic solution of the ZBW equation of motion (13) endowed with a given frequency ω can be put as

$$v^\mu(\tau) = \frac{p^\mu}{M} + a^\mu \cos(\omega\tau) + b^\mu \sin(\omega\tau) \tag{24}$$

Inserting this equation in the ZBW equation

$$v^\mu \;=\; \frac{p^\mu}{M} - \sum_{i=1}^{n} (-1)^i k_i v_\mu^{(2i)} \tag{25}$$

which, reminding that $k_0 = M$, can be re-written as

$$\sum_{i=0}^{n} (-1)^i k_i v_\mu^{(2i)} \;=\; \frac{p^\mu}{M}\,, \tag{26}$$

we get the "characteristic" equation in the unknown ω

$$\sum_{i=0}^{n} k_i \omega^{2i} \;=\; 0 \tag{27}$$

In order that for any solution of the previous equation be

$$\omega_i^2 = \omega_0^2 \qquad\qquad i = 1, \ldots n$$

the l.h.s. of (27) must be proportional to the polynomial F_n of grade n in the variable ω^2 defined as follows

$$F_n \equiv \sum_{i=0}^{n} \lambda_i \omega^{2i} \equiv (\omega^2 - \omega_0^2)^n \tag{28}$$

where λ_i are expressed through the Newton binomial coefficients

$$\lambda_i \;=\; \frac{(-1)^{n-i}n!}{i!(n-i)!}\omega_0^{2(n-i)} \tag{29}$$

For the condition $\sum_{i=0}^{n} k_i\omega^{2i} \propto F_n$ we have to require

$$k_i \propto \lambda_i \tag{30}$$

Since $k_0 \equiv M$ and $\lambda_0 = (-1)^n\omega_0^{2n}$, from (30) we infer

$$k_i \;=\; \frac{(-1)^n M}{\omega_0^{2n}}\lambda_i = \frac{(-1)^i n!}{i!(n-i)!}\frac{M}{(\omega_0)^{2i}} \tag{31}$$

On inserting solution (24) with $\omega = \omega_0$ and coefficients k_i given by Eq. (31) in ($n \geq 1$) spin vector formula (22)

$$\boldsymbol{s} = \sum_{i=1}^{n} k_i \sum_{l=0}^{i-1}(-1)^{i-l-1}\boldsymbol{v}^{(l)} \times \boldsymbol{v}^{(2i-l-1)} \tag{32}$$

and referring to CMF ($\boldsymbol{p} = 0$), after some algebra we obtain the spin vector in the following form

$$\boldsymbol{s} = \sum_{i=1}^{n} \frac{(-1)^i n!}{(i-1)!(n-i)!} m\boldsymbol{r} \times \boldsymbol{v} \tag{33}$$

This expression is different from 0 only for $n = 1$ ($\mathrm{NNM}^{(1)}$, i.e.: the Dirac case), whilst for $n \geq 2$ we identically get

$$\boldsymbol{s} \;=\; 0 \tag{34}$$

Therefore, when the intrinsic —i.e. in the CMF— motion is harmonic[11] and is endowed with a degenerate frequency, the spin vector does vanish (only) in CMF, so that *helicity and intrinsic angular momentum are zero.* [12]

[11] For example when it is uniform circular and lightlike with $v^2 = 0 =$ constant: cf. the classification of ZBW motions proposed in [13].

[12] Note that even if the spin is zero in the CMF, the intrinsic magnetic momentum $\frac{1}{2}e\boldsymbol{r} \times \boldsymbol{v}$ does not indentically vanish.

Let us notice a quite interesting analogy between the degenerate frequency of the above seen ZBW inner motion and the degenerate oscillations ($\omega_1 = \omega_2$) found in pure fourth-order case of the Pais-Uhlenbeck double oscillator [15]. As a matter of fact, in [16] it is proved that the degenerate Pais-Uhlenbeck scalar field can be identified with the helicity-0 longitudinal component of the Stückelberg field.

In addition to the special case of motion with a $n-$degenerate frequency studied above,in [13] we predicted particles endowed with non-zero spin vector and ZBW in a generic frame, but with zero intrinsic angular momentum and helicity, for any $n \geq 1$ NNM$^{(n)}$ *and for any given coefficients* k_i. It is sufficient to consider, for a chosen $\mathcal{L}^{(n)}$, those solutions of the Euler-Lagrange equation which entail a *rectilinear oscillatory motion* in the CMF. To make an example, for $\mathcal{L}^{(1)}$ it is enough to assume, whichever is the chosen value of k_1, a rectilinear harmonic[13] motion among the solutions of the ZBW equation (13)

$$v^\mu = \frac{p^\mu}{m} + F^\mu \cos(\omega\tau) \tag{35}$$

The space trajectory turns out to be a tilted sinusoid-like path belonging to the plane α containing (as usual we label by $_\star$ the quantities referred to the CMF) $\boldsymbol{v}_\star$, $\boldsymbol{a}_\star$ and $\boldsymbol{p}$, with the nodal axis parallel to $\boldsymbol{p}$. Actually the trajectory is quite different from the one of a spinless Klein–Gordon particle (described by NNM$^{(0)}$), which is a straight line. The spin vector is equal to $k_1\,(\boldsymbol{v}\times\boldsymbol{a})$ because of Eq. (18). By labelling with $\perp$ the component orthogonal to the boost and taking into account that, since the Lorentz boost $-\boldsymbol{w}$[14] does not affect the components orthogonal to $\boldsymbol{p}$, $\boldsymbol{F}\times\boldsymbol{p} = \boldsymbol{F}^\perp\times\boldsymbol{p} = \boldsymbol{F}^\perp_\star\times\boldsymbol{p}$, we obtain

$$\boldsymbol{s} \;=\; \frac{1}{\sqrt{\omega}}\,(\boldsymbol{F}^\perp_\star\times\boldsymbol{p})\,\sin(\omega\tau)\,. \tag{36}$$

[13]The linear oscillatory motion is harmonic only for $\mathcal{L}^{(1)}$; it is in general anharmonic for $n \geq 2$.

[14]As made in Sect. 1, we apply an arbitrary boost $-\boldsymbol{w}$ to the CMF and consequently the particle's center-of-mass, at rest in the CMF, acquires a velocity $\boldsymbol{w}$ and a momentum $\boldsymbol{p} = p^0\boldsymbol{w} = \gamma m\boldsymbol{w}$ in the new frame.

Actually the spin does not preceed anymore as the spin of a Dirac particle does, but *linearly vibrates along a direction orthogonal to the momentum*, and the helicity is zero. As expected, in the CMF, where $\boldsymbol{p} = 0$, the spin vector identically vanishes, $\boldsymbol{s}_\star = 0$; while in a generic frame it vanishes only after averaging over a ZBW period, $\overline{\boldsymbol{s}} = 0$ (analogously for HZPs it vanishes the quantum average spin, according to the Correspondence Principle). As is easy to check, from eqs. (19-20) and from the analogous formulae for $n > 3$, also for $n > 1$ in correspondence to CMF rectilinear motions the intrinsic angular momentum $\boldsymbol{s}_\star$ does vanish at any time, since $\boldsymbol{v}_\star$ and its time derivatives are collinear vectors ($\boldsymbol{v}_\star // \boldsymbol{a}_\star // \dot{\boldsymbol{a}}_\star // \ \ldots$). But it easily proved that after a boost the Lorentz-transformed spin vector $\boldsymbol{s}$ is in general non-zero. In fact, labelling with $\parallel$ ($\perp$) the components parallel (orthogonal) to the boost $-\boldsymbol{w}$, and taking into account that $v^0_\star \equiv \dfrac{\mathrm{d}\tau}{\mathrm{d}\tau} = 1$ and $a^0_\star = 0$[15] we can write for the Lorentz-transformed components of the 4-vectors v^μ and a^μ

$$v^{\parallel} = \gamma\,(v^{\parallel}_\star + w) \qquad\qquad v^{\perp} \;=\; v^{\perp}_\star\,,$$

$$a^{\parallel} = \gamma\, a^{\parallel}_\star \qquad\qquad a^{\perp} \;=\; a^{\perp}_\star\,,$$

and so on for the higher-order derivatives of the velocity. As a consequence, in the new frame the transformed velocity $\boldsymbol{v}$ and its transformed derivatives *are not anymore collinear*, so that $\boldsymbol{s} \neq 0$. It is also easy to check that $\boldsymbol{v}$, $\boldsymbol{a}$, etc. belong to the aforesaid plane α. Being orthogonal to the plane α for Eqs. (18)-(20) and for the analogous formulae for $n > 3$, the spin is then normal to the momentum, so that, as expected, the helicity always vanishes.

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[15] In the CMF we can write $v^\mu = (1;\, \boldsymbol{v}_\star)$, $a^\mu = (0;\, \boldsymbol{a}_\star)$.

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SOLUTION OF CERTAIN GENERALIZED FRACTIONAL KINETIC EQUATIONS

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Abstract

In view of the usefulness and importance of fractional kinetic equations in applied problems of science, engineering and technology, the authors present the solutions of certain fractional kinetic equations which provide extensions of the earlier derived results on the fractional kinetic equations solved by Nonnemacher and Metzler (1995), Saxena, Mathai and Haubold (2002, 2004, 2006). The method of derivation of the solution essentially depends on the use of Riemann-Liouville fractional calculus operators. It has been shown that by the application of Riemann-Liouville fractional integral operator and its interesting properties, the solution of the given fractional kinetic equation can be obtained in a straight forward manner. This method does not make use of the classical Laplace transform. The results presented are in compact form and convenient for numerical computation. Furthermore some figures to illustrate the behavior of the solution are given.

Keywords: Fractional kinetic equation, Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Mittag-Leffler function, generalized Mittag-Leffler function.
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1 Introduction

Fractional kinetic equations have gained importance due their wide range of applications in problems arising in science, engineering and technology. In this connection, one can refer to the monographs by Podlubny [14], and Kilbas et al. [8] and the various papers given therein. Fractional kinetic equations are studied by Zaslavasky [26], Glöckle and Nonnenmacher [5], Nonnenmacher and Metzler [12], Saxena et al. ([17]-[19]) and several others due to their importance in the solution of certain applied problems governing reaction and relaxation in complex systems, anomalous diffusion etc. This method presents a new simple method for determining the solution of generalized forms of the kinetic equations discussed earlier by Saxena et al. [18, 19]. The Mittag-Leffler function naturally occurs as a solution of fractional order differential equation or a fractional order integral equation. The Mittag-Leffler function [11] is defined in the literature in terms of the power series:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \mathrm{Re}(\alpha) > 0. \tag{1.1}$$

This function is generalized by Wiman [25] in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0. \tag{1.2}$$

According to Dzherbashyan [2, p.118; also see 3] both the functions defined by the equations (1.1) and (1.2) are entire functions of order $\rho = \frac{1}{\alpha}$ and type $\sigma = 1$. A comprehensive and detailed account of these functions is available from the monograph of Erdélyi et al. [4, chapter 18], Dzherbashyan [2, Chapter 2] and a survey paper by Haubold et al. [7].

Prabhakar [15] introduced a very useful generalization of (1.2) in the following interesting form

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\, n!} \tag{1.3}$$

where $\gamma \in C, \gamma \neq 0, -1, -2, \ldots, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0,$ *and*

$$(\gamma)_n = \begin{cases} 1 & if \ \ n = 0 \\ \gamma(\gamma+1)\ldots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} & if \ \ n = 1, 2, 3, \ldots. \end{cases} \tag{1.4}$$

The Riemann-Liouville fractional integral operators are defined in the books of Miller and Ross [10], Oldham and Spanier [13] and Podlubny [14].

The Riemann-Liouville fractional integral operator of $f(t)$ of order ν is defined as [13; also see 8-10]

$${}_aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-u)^{\nu-1} \, f(u)du \tag{1.5}$$

where $\mathrm{Re}(\nu) > 0, t > a;$ with ${}_aD_t^0 f(t) = f(t).$

The Riemnn-Liouville fractional derivative of $f(t)$ of order μ is defined as [3]

$${}_aD_t^{\mu} f(t) = \frac{d^n}{dt^n} \left({}_aD_t^{\mu-n} f(t) \right), \tag{1.6}$$

where $\mathrm{Re}(\mu) \geq 0, t > a, n = [\mu] + 1.$

The following properties hold [14]

i) If $f(t)$ is continuous for $t \geq a$ then

$${}_aD_t^{-p}\left({}_aD_t^{-q} f(t) \right) = {}_aD_t^{-p-q} f(t). \tag{1.7}$$

ii) If the fractional derivative ${}_aD_t^q f(t)$ of a function $f(t)$ is integrable, then

$$ {}_aD_t^{-p}\left({}_aD_t^q f(t)\right) = {}_aD_t^{q-p} f(t) \sum_{j=1}^{k} \left[{}_aD_t^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}, \qquad (1.8) $$

with $k-1 \leq p < k$.

iii) $$ {}_aD_t^{p}\left({}_aD_t^{-q} f(t)\right) = {}_aD_t^{p-q} f(t), \qquad (1.9) $$

where we assume that $f(t)$ is continuous for $t \geq a$ and, if $p \geq q \geq 0$, that the derivative ${}_aD_t^{p-q} f(t)$ exists.

By virtue of the definitions (1.5) and (1.6), it is not difficult to show that

$$ {}_aD_t^{-\nu}(t-a)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+\nu)}(t-a)^{\rho+\nu-1}, \qquad (1.10) $$

where $\mathrm{Re}(\rho) > 0, \mathrm{Re}(\nu) > 0, t > a$; and

$$ {}_aD_t^{\nu}(t-a)^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-\nu)}(t-a)^{\rho-\nu-1}, \qquad (1.11) $$

where $\mathrm{Re}(\rho) > 0, \mathrm{Re}(\nu) > 0, t > a.$

The object of this paper is to establish some theorems for the derivation of the solution of fractional kinetic equation in which Riemann-Liouville fractional integral ${}_0D_t^{-\nu}$ is replaced by ${}_aD_t^{-\nu}$. The method is different from the earlier method in which Laplace transform has been used. This method neither uses the Laplace transform nor the Sumudu transform. The special cases of the theorems includes, as special cases results given earlier by Glockle and Nonnenmacher [5], Saxena et al. [23] , Saxena and Kalla [16], Shafee et al. [24], and others. The results are obtained in terms of generalized Mittag-Leffler functions and hypergeometric function. Furthermore some figures to illustrate the behavior of the solution are given.

2 Fractional Kinetic Equation with Riemann-Liouville Fractional Integral ${}_aD_t^{-\nu}$

Theorem 2.1 If min {Re(c), Re(ν)} > 0, then for the fractional kinetic equation

$$N(t) - N_a f(t) = -c^{\nu}\, {}_aD_t^{-\nu} N(t), \tag{2.1}$$

where $N_a > 0$ is a constant, $f(t)$ is any integrable function on the finite interval $[a, b]$, exists a unique solution of (2.1) given by

$$N(t) = N_a \sum_{m=0}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} f(t). \tag{2.2}$$

Proof: If we multiply both sides of (2.1) by $(-c^{\nu})^m \, {}_aD_t^{-m\nu}$ and we use (1.7), we obtain

$$(-c^{\nu})^m \, {}_aD_t^{-m\nu} N(t) - (-c^{\nu})^{m+1} \, {}_aD_t^{-m\nu-\nu} N(t) =$$

$$N_a (-c^{\nu})^m \, {}_aD_t^{-m\nu} f(t), \quad (m \epsilon N_0). \tag{2.3}$$

Now summing up both sides of (2.3) for m from 0 to ∞, it yields

$$\sum_{m=0}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} N(t) - \sum_{m=0}^{\infty} (-c^{\nu})^{m+1} \, {}_aD_t^{-m\nu-\nu} N(t) = N_a \sum_{m=0}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} f(t)$$

which can be written as

$$\sum_{m=0}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} N(t) - \sum_{m=1}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} N(t) = N_a \sum_{m=0}^{\infty} (-c^{\nu})^m \, {}_aD_t^{-m\nu} f(t),$$

which on cancelation of the common terms becomes

$$N(t) = N_a \sum_{m=0}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu} f(t). \tag{2.4}$$

This completes the proof of Theorem 2.1.

Note 1: We note that the method adopted in deriving the solution of (2.1) is similar to that used by Al-Saqabi and Tuan [1].

Note 2: The solution of fractional reaction-diffusion equations are derived by Saxena et al. ([20]-[22]).

Verification of the solution: In order to verify the solution, we multiply (2.4) by $(c^{\nu})\ {}_aD_t^{-\nu}$, then it becomes

$$c^{\nu}\ {}_aD_t^{-\nu} N(t) \ = \ -N_a \sum_{m=0}^{\infty}(-c^{\nu})^{m+1} \ {}_aD_t^{-m\nu-\nu} f(t); \ \ m\epsilon N_0 . \tag{2.5}$$

From (2.5), we find that

$$N(t) + c^{\nu}\ {}_aD_t^{-\nu} N(t) = N_a \sum_{m=0}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu} f(t) -$$

$$N_a \sum_{m=0}^{\infty}(-c^{\nu})^{m+1} \ {}_aD_t^{-(m+1)\nu} f(t); \tag{2.6}$$

$$= N_a \sum_{m=0}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu} f(t) - N_a \sum_{m=1}^{\infty}(-c^{\nu})^m \ {}_aD_t^{-m\nu} f(t) \tag{2.7}$$

$$= N_a f(t).$$

This shows that (2.2) is the unique solution of (2.1). This completes the proof of Theorem 2.1.

Special cases. For $f(t)$=1, theorem 2.1 reduces to the following result given by Saxena et al. [23]:

Corollary 2.1 If Re(c) $>$ 0, Re(ν) $>$ 0, then the unique solution of the fractional kinetic equation

$$N(t) - N_a = -c^{\nu}\ {}_aD_t^{-\nu}N(t) \tag{2.8}$$

is given by

$$N(t) = N_a E_{\nu}(-c^{\nu}(t-a)^{\nu}),\ t > a \tag{2.9}$$

where $E_{\nu}(t)$ is the Mittag-Leffler function defined by (1.1).

(2.9) follows from (2.2) by using the formula (1.10).

Corollary 2.2 If Re(c) $>$ 0, Re(ν) $>$ 0, then the unique solution of the fractional kinetic equation

$$N(t) - N_0 = -c^{\nu}\ {}_0D_t^{-\nu}N(t) \tag{2.10}$$

is given by

$$N(t) = N_0 E_{\nu}(-c^{\nu}t^{\nu}),\ \ t > 0 \tag{2.11}$$

where $E_{\nu}(t)$ is the Mittag-Leffler function defined by (1.1). The above Corollary has been given by Haubold and Mathai [6].

If we set $f(t) = (t-a)^{\mu-1}$, then using the result (1.10) we arrive at the following

Corollary 2.3: If min{Re(μ), Re(ν)} $>$ 0, Re(c) $>$ 0, then the solution of the fractional kinetic equation

$$N(t) - N_a\ (t-a)^{\mu-1} = -c^{\nu}\ {}_aD_t^{-\nu}N(t), \tag{2.12}$$

is given by

$$N(t) = N_a\,\Gamma(\mu)\,(t-a)^{\mu-1}E_{\nu,\mu}(-c^{\nu}(t-a)^{\nu}),\ t>a \tag{2.13}$$

where $E_{\nu,\mu}(t)$ is the generalized Mittag-Leffler function defined by (1.2).

As a tends to zero, (2.13) reduces to the following result given by Saxena and Kalla [16]:

Corollary 2.4: If $\min\{\mathrm{Re}\,(\mu), \mathrm{Re}(\nu)\} > 0, \mathrm{Re}(c) > 0,$ then the solution of the fractional kinetic equation

$$N(t) - N_0\,t^{\mu-1} = -c^{\nu}\ {}_0\mathrm{D}_t^{-\nu}N(t), \tag{2.14}$$

is given by

$$N(t) = N_0\,\Gamma(\mu)\,t^{\mu-1}E_{\nu,\mu}(-c^{\nu}t^{\nu}),\ t>0 \tag{2.15}$$

where $E_{\nu,\mu}(t)$ is the generalized Mittag-Leffler function defined by (1.2).

3 Generalized Fractional Kinetic Equation

Now we generalize the fractional kinetic equation with Riemann-Liouville fractional integral ${}_a\mathrm{D}_t^{-\nu}$ given in the previous section.

Theorem 3.1 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0,\ \mathrm{Re}(c) > 0,$ then for the fractional kinetic equation

$$c^{\mu}\ {}_a\mathrm{D}_t^{-\mu}N(t) - N_a f(t) = -c^{\nu}\ {}_a\mathrm{D}_t^{-\nu}N(t), \tag{3.1}$$

where $N_a > 0$ is a constant, $f(t)$ is any integrable function on the finite interval $[a, \mathrm{b}]$, exists a unique solution of (3.1) given by

$$N(t) = \frac{N_a}{c^{\mu}} \sum_{m=0}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{\mu}\left({}_aD_t^{-(\nu-\mu)m} f(t)\right). \qquad (3.2)$$

Proof: Applying the operator $(-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m}$, $m \in N_0$, both sides of (3.1) and using (1.7), we obtain

$$c^{\mu} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\mu} N(t) + c^{\nu} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\nu} N(t) =$$

$$N_a (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m} f(t),$$

which can be written as

$$c^{\mu} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\mu} N(t) - c^{\mu} (-c^{\nu-\mu})^{m+1} \; {}_aD_t^{-(\nu-\mu)m-\nu} N(t)$$

$$= N_a (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m} f(t). \qquad (3.3)$$

Now summing up both sides of (3.3) for m from 0 to ∞, it yields

$$c^{\mu} \sum_{m=0}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\mu} N(t) -$$
$$c^{\mu} \sum_{m=0}^{\infty} (-c^{\nu-\mu})^{m+1} \; {}_aD_t^{-(\nu-\mu)m-\nu} N(t) =$$
$$N_a \sum_{m=0}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m} f(t),$$

that is,

$$c^{\mu} \sum_{m=0}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\mu} N(t) -$$
$$c^{\mu} \sum_{m=1}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m-\mu} N(t) =$$
$$N_a \sum_{m=0}^{\infty} (-c^{\nu-\mu})^m \; {}_aD_t^{-(\nu-\mu)m} f(t),$$

and canceling the common terms we get

$$c^{\mu}\ {}_aD_t^{-\mu}N(t) = N_a\ \sum_{m=0}^{\infty}(-c^{\nu-\mu})^m\ {}_aD_t^{-(\nu-\mu)m}f(t).$$

Then applying the operator ${}_aD_t^{\mu}$ and by virtue of (1.9) we obtain

$$N(t) = \frac{N_a}{c^{\mu}}\ \sum_{m=0}^{\infty}(-c^{\nu-\mu})^m\ {}_aD_t^{\mu}\left({}_aD_t^{-(\nu-\mu)m}f(t)\right).$$

This completes the proof of Theorem 3.1.

Note 3: From (3.2) using the property (1.9) results

$$N(t) = \frac{N_a}{c^{\mu}}\ \sum_{m=0}^{\infty}(-c^{\nu-\mu})^m\ {}_aD_t^{\mu-(\nu-\mu)m}f(t),$$

and if ${}_aD_t^{\mu}f(t)$ is integrable and $\left[{}_aD_t^{\mu-j}f(t)\right]_{t=a} = 0,\ j = 1, 2, \cdots k,$ then from (1.8)

$$N(t) = \frac{N_a}{c^{\mu}}\ \sum_{m=0}^{\infty}(-c^{\nu-\mu})^m\ {}_aD_t^{-(\nu-\mu)m}\left({}_aD_t^{\mu}f(t)\right). \tag{3.4}$$

This result when $a \to 0^+$ becomes

$$N(t) = \frac{N_0}{c^{\mu}}\ \sum_{m=0}^{\infty}(-c^{\nu-\mu})^m\ {}_0D_t^{-(\nu-\mu)m}\left({}_0D_t^{\mu}f(t)\right),$$

that is,

$$N(t) = \frac{N_0}{c^{\mu}}\left[{}_0D_t^{\mu}f(t) + \sum_{m=1}^{\infty}(-c^{\nu-\mu})^m\ {}_0D_t^{-(\nu-\mu)m}\left({}_0D_t^{\mu}f(t)\right)\right], \tag{3.5}$$

which is a know result [24].

Special Cases: Let be $f(t) = (t-a)^{\delta-1}$, then we get the following

Corollary 3.1 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re(c)} > 0$, then the solution of the fractional kinetic equation

$$c^{\mu}\ {}_{a}\mathrm{D}_{t}^{-\mu}N(t) - N_{a}\ (t-a)^{\delta-1} = -c^{\nu}\ {}_{a}\mathrm{D}_{t}^{-\nu}N(t), \tag{3.6}$$

is given by

$$N(t) = \frac{N_a}{c^{\mu}}\ \Gamma(\delta)\ (t-a)^{\delta-\mu-1}\ E_{\nu-\mu,\delta-\mu}(-c^{\nu-\mu}(t-a)^{\nu-\mu}),\ t > a. \tag{3.7}$$

Proof. Using (1.10)

$${}_{a}\mathrm{D}_{t}^{-(\nu-\mu)\mathrm{m}}(\mathrm{t}-a)^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+(\nu-\mu)m)}(\mathrm{t}-a)^{\delta+(\nu-\mu)m-1},$$

and applying (1.11)

$${}_{a}\mathrm{D}_{t}^{\mu}\left({}_{a}\mathrm{D}_{t}^{-(\nu-\mu)\mathrm{m}}(\mathrm{t}-a)^{\delta-1}\right) = \frac{\Gamma(\delta)}{\Gamma(\delta+(\nu-\mu)m-\mu)}(\mathrm{t}-a)^{\delta+(\nu-\mu)m-\mu-1}.$$

Then from (3.2) we get

$$N(t) = \frac{N_a}{c^{\mu}}\ \Gamma(\delta)\ (t-a)^{\delta-\mu-1}\ \sum_{m=0}^{\infty}\frac{(-1)^{\mathrm{m}}\ (c(t-a))^{(\nu-\mu)m}}{\Gamma(\delta+(\nu-\mu)m-\mu)},$$

and from (1.2)

$$N(t) = \frac{N_a}{c^{\mu}}\ \Gamma(\delta)\ (t-a)^{\delta-\mu-1}\ E_{\nu-\mu,\delta-\mu}(-c^{\nu-\mu}(t-a)^{\nu-\mu}).$$

Corollary 3.2 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0,\ \mathrm{Re}(1-\mu) > 0, \mathrm{Re(c)} > 0$, then the solution of the fractional kinetic equation

$$c^{\mu}\ {}_{a}D_{t}^{-\mu}N(t) - N_{a} = -c^{\nu}\ {}_{a}D_{t}^{-\nu}N(t), \tag{3.8}$$

is given by

$$N(t) = \frac{N_a}{c^{\mu}}\ (t-a)^{-\mu}\ E_{\nu-\mu,1-\mu}(-c^{\nu-\mu}(t-a)^{\nu-\mu}),\ t > a. \tag{3.9}$$

Corollary 3.3 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re(c)} > 0$, then the solution of the fractional kinetic equation

$$c^{\mu}\ {}_{0}D_{t}^{-\mu}N(t) - N_0\ t^{\delta-1} = -c^{\nu}\ {}_{0}D_{t}^{-\nu}N(t) \tag{3.10}$$

is given by

$$N(t) = \frac{N_0}{c^{\mu}}\ \Gamma(\delta)\ t^{\delta-\mu-1}\ E_{\nu-\mu,\delta-\mu}(-c^{\nu-\mu}\ t^{\nu-\mu}),\ t > 0. \tag{3.11}$$

If $f(t) = (t-a)^{\delta-1}\ e^{-d(t-a)} = \sum_{k=0}^{\infty} \frac{(-d)^k}{k!}(t-a)^{k+\delta-1}$, then we have

Corollary 3.4 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re(c)} > 0$, then the solution of the fractional kinetic equation

$$c^{\mu}\ {}_{a}D_{t}^{-\mu}N(t) - N_a\ (t-a)^{\delta-1}\ e^{-d(t-a)} = -c^{\nu}\ {}_{a}D_{t}^{-\nu}N(t), \tag{3.12}$$

is given by

$$N(t) = \frac{N_a}{c^{\mu}}\ \Gamma(\delta)\ (t-a)^{\delta-\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m (c(t-a))^{(\nu-\mu)m}}{\Gamma(\delta+(\nu-\mu)m-\mu)} \quad \text{x}$$

$$ {}_{1}F_{1}\big(\delta; \delta + (\nu-\mu)m - \mu; -\mathrm{d}(t-a)\big),\ t > a. \tag{3.13}$$

Proof. From (1.10)

$${}_aD_t^{-(\nu-\mu)m}(t-a)^{\delta-1}e^{-d(t-a)} =$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(\delta+k)}{\Gamma(\delta+k+(\nu-\mu)m)} \frac{(-d)^k}{k!} (t-a)^{\delta+k+(\nu-\mu)m-1}$$

and the application of (1.11) leads us to

$${}_aD_t^{\mu}\left({}_aD_t^{-(\nu-\mu)m}(t-a)^{\delta-1}e^{-d(t-a)}\right) =$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(\delta+k)}{\Gamma(\delta+k+(\nu-\mu)m-\mu)} \frac{(-d)^k}{k!} (t-a)^{\delta+k+(\nu-\mu)m-\mu-1},$$

which can be written as

$${}_aD_t^{\mu}\left({}_aD_t^{-(\nu-\mu)m}(t-a)^{\delta-1}e^{-d(t-a)}\right) =$$

$$\frac{\Gamma(\delta)}{\Gamma(\delta+(\nu-\mu)m-\mu)} (t-a)^{\delta+(\nu-\mu)m-\mu-1} \quad x$$

$${}_1F_1\big(\delta;\delta+(\nu-\mu)m-\mu;-d(t-a)\big).$$

Therefore (3.2) take the form

$$N(t) = \frac{N_a}{c^{\mu}} \Gamma(\delta)\,(t-a)^{\delta-\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m (c(t-a))^{(\nu-\mu)m}}{\Gamma(\delta+(\nu-\mu)m-\mu)} \quad x$$

$${}_1F_1\big(\delta;\delta+(\nu-\mu)m-\mu;-d(t-a)\big).$$

Corollary 3.5 If $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re(c)} > 0$, then the solution of the fractional kinetic equation

$$c^{\mu}\ {}_0D_t^{-\mu}N(t) - N_0\ t^{\delta-1}\ e^{-dt} = -c^{\nu}\ {}_0D_t^{-\nu}N(t), \qquad (3.14)$$

is given by

$$N(t) = \frac{N_0}{c^{\mu}}\ \Gamma(\delta)\ t^{\delta-\mu-1} \quad x$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m (ct)^{(\nu-\mu)m}}{\Gamma(\delta+(\nu-\mu)m-\mu)} \quad {}_1F_1(\delta; \delta + (\nu-\mu)m - \mu; -dt), \quad t > 0. \qquad (3.15)$$

The results (3.11) and (3.15) were given by Shafee et al. [24].

Following two figures are presented, which show the behavior of N(t) for $f(t) = (t-a)^{\delta-1}$ and $f(t) = (t-a)^{\delta-1}\ e^{-d(t-a)}$, for selected values of the parameters.

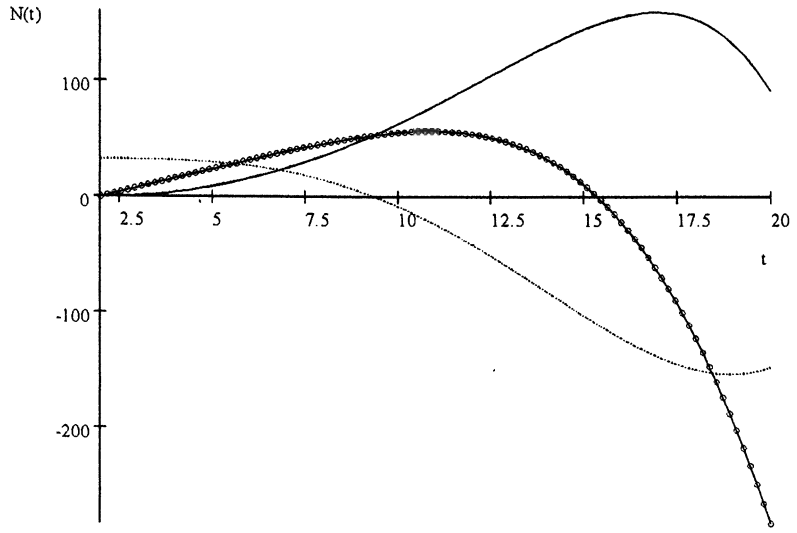

Figure 1 $N(t)$ for $f(t) = (t-a)^{\delta-1}$ with $N_a = 1, c = 0.25, \delta = 3, \nu = 5, a = 2$ and different values of μ

_____ $\mu = 0$ —o—o—o— $\mu = 1$ -------- $\mu = 2$

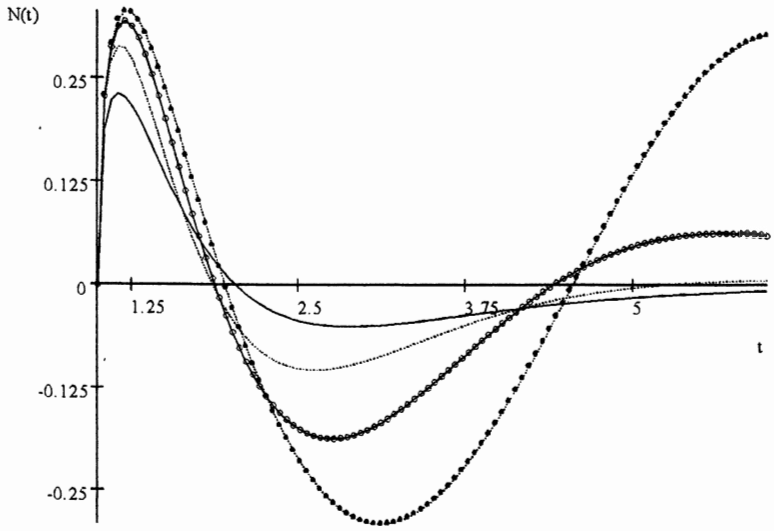

Figure 2 $N(t)$ for $f(t)=(t-a)^{\delta-1}\,e^{-d(t-a)}$ with $N_a=1=c,\ \delta=2,\ \mu=1/2,\ a=1,\ d=3/2$ and different values of ν

$\nu=1$ -------- $\nu=3/2$ $\nu=2$ $\nu=5/2$

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GRAVITY OF ACCELERATIONS ON QUANTUM SCALES AND ITS CONSEQUENCES

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Abstract

Gravity stands apart from other fundamental interactions in that it is locally equivalent to an accelerated frame and can be transformed away. Again it is indistinguishable from the geometry of space-time (which is an arena for all other basic interactions), its strength being linked with the curvature. This is a major reason why it has so far not been amenable to quantisation like other interactions. It is also evident that new ideas are required to resolve several conundrums in areas like cosmology, black hole physics, and particles at high energies. That gravity can have strong coupling at microscales has also been suggested in several contexts earlier. Here we develop some of these ideas, especially in connection with the high accelerations experienced by particles at microscales, which would be interpreted as strong local gravitational fields. The consequences are developed for various situations and possible experimental manifestations are discussed.

Keywords: Strong gravity; Effective gravitational constant; Spiral trajectories; Acceleration at microscales.

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1 Introduction

Gravity is one of the four fundamental interactions underlying the plethora of physical phenomena describing forces between particles and fields in space time. It is distinct from other forces like electromagnetism or nuclear in that it is locally equivalent to an accelerated frame and is described by the curvature of space time. Moreover it couples universally to all other fields. So gravity stands apart from other interactions and its unification with other forces is yet to be accomplished. A basic tenet of general relativity is the equivalence principle, i.e. a gravitational field is locally equivalent to an accelerated frame. Lorentz invariance corresponds to Inertial frames (IF's), moving with uniform velocity or at rest.

2 Acceleration at microscales and strong gravity

They are associated with absence of forces (or accelerations) and are described in flat, Minkowskian space time (no gravitational field). Inertial frames are an idealization. In practice subatomic particles feel forces such as electromagnetic or strong interactions over quantum scales of length and time. A particle initially at rest could be accelerated to high velocities over short time scale, i.e. over $\sim 10^{-16} - 10^{-23} s$. Typically this acceleration occurs over a time scale of $t \sim \hbar / mc^2$ (for a particle of mass m), so that acceleration is $\sim \frac{c}{\hbar / mc^2}$, i.e.

$$a \sim \frac{mc^3}{\hbar} \tag{1}$$

For $m \sim 10^{-27} kg$ (proton mass), this corresponds to $a \sim 10^{32} m/s^2$.

So by equivalence principle this should correspond to a 'strong gravitational field', over a distance scale $\sim \hbar / mc$. For two protons, with this strong force (accelerating them) causing this acceleration, *a*, their equivalent mutual gravitational interaction is given by:

$$a \sim \frac{mc^3}{\hbar} \approx \frac{G_{eff} m}{r^2} \tag{2}$$

This implies on 'effective gravitational constant' of

$$G_{eff} \approx \frac{\hbar c}{m^2} \approx 10^{38} G_{Newton} \tag{3}$$

over a distance scale of $\sim \hbar / mc$.

Once the protons leave the interaction (acceleration) region they would have uniform velocity ~c, so that once again we have IF and Lorentz invariance. Thus the time scale for presence of a non IF (acceleration) is $\hbar / mc^2 \approx 10^{-23} s$.

So this high acceleration is equivalent to a strong gravitational field with an effective $G \sim 10^{38} G_{Newton}$ over a distance scale $\sim \hbar / mc$.

Indeed this picture provides an additional framework for the strong gravity hypothesis (in considering particle interactions in the micro world) proposed several years ago.

For a detailed description of several interesting consequences of the approach see the extensive review article [1, 2] and references therein and more recently the chromo-gravity approach [3, 4, 5]. The past literature on this is extensive [6, 7, 8, 9, 10].

In this picture the short range hadronic gravitational interactions are mediated by massive spin-2 mesons (same quantum numbers as graviton). Starting with linear spin-2 field equation and successively adding all non-linear self interactions (of a non-abelian field line gravity) one does get Einstein type field equations with large curvature and strong coupling.

More recently [11] we have the ADD conjecture that gravity becomes strong at electroweak scale, giving an extra dimension of few microns. Gravity weakens much faster in higher dimensions. Hence we have an exponential short range gravitational field with large coupling (with large acceleration at the quark scale being the underlying feature).

Indeed from the Riemann equation:

$$a^{\mu} = R^{\mu}_{\alpha\beta\gamma} u^{\alpha} u^{\beta} n^{\gamma} \tag{4}$$

this large acceleration is also equivalent to a high curvature (over a subatomic scale), i.e.

$$R \sim \left(\frac{mc}{\hbar} \right)^2 \approx 10^{24} m^{-2} \tag{5}$$

A bunch of such particles in the high acceleration region (when they are being accelerated over quantum distance and time scales) would also experience a temperature

$$T \sim \frac{\hbar a}{2\pi k_B c} \tag{6}$$

This implies

$$T \sim \frac{mc^2}{2\pi k_B} \approx 2 \times 10^{12} K \tag{7}$$

Indeed in high energy collisions, there is a transverse thermal distribution of particles with this temperature, corresponding to the so called Hagedorn temperature. [1, 2]

3 Effective gravitational constant and modification of General Relativity

As the upper limit to particle mass m is the Planck mass $m_{Pl} \sim \left(\frac{\hbar c}{G}\right)^{1/2} \approx 10^{19} GeV$ this implies a maximum acceleration of: [12]

$$a_{\max} \sim \frac{m_{Pl} c^3}{\hbar} \approx 10^{51} m/s^2 \tag{8}$$

Corresponding to a highest curvature $\approx c^3/\hbar G \approx 10^{61} m^{-2}$ with a maximum temperature:

$$T_{\max} \sim \frac{m_{Pl} c^2}{2\pi k_B} \approx 10^{32} K \tag{9}$$

These upper limits (arising from quantum constraints on space time) could ameliorate classical singularities where, as $T \to \infty$, $R \to \infty$ as in the big bang singularity or in black hole collapse to a singular state [24, 25]. The classical GR could undergo modifications due to these limitations.

In this connection it is interesting that the use of minimum velocity [13], $\bar{c} \approx 3 \times 10^{-11} cm/s$, gives $a \approx 10^{-8} cm/s^2$, the MOND value for acceleration, when used in equation (1).

In another context we had suggested a Born – Infeld type of modification of GR, with a maximum curvature $R_{\max}$, with a modified action like $\left(\frac{R}{1 - R/R_{\max}}\right)\sqrt{-g}$, reducing to usual GR for $R << R_{\max} \approx L_{Pl}^{-2}$. [14, 15]

In the above examples involving equation (2) and (3), we had an effective gravitational constant G_{eff}. Following a Sakharov type picture, we can write this as: [15, 16]

$$G_{eff} \approx \frac{c^5}{\hbar \int_0^{\omega_c} \omega d\omega} \tag{10}$$

(which for $\omega_c \approx \frac{mc^2}{\hbar}$, gives $G_{eff} \sim \frac{\hbar c}{m^2}$ as above).

For the highest energy interactions, $\omega_c \approx \frac{m_{Pl} c^2}{\hbar}$ and this gives $G = G_{eff}$!

The action can be expanded in powers of the curvature, with a $R_{\max} \approx \left(\frac{mc}{\hbar}\right)^2$. [16]

Essentially the effective gravity constant arises from all the zero point fluctuations of the various quantum fields (in curved space), which for an inertial frame has energy density:

$$\rho(\omega)d\omega \approx \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{2}d\omega \tag{11}$$

and for an accelerated frame this gets modified to:

$$\rho(\omega)d\omega \approx \frac{\omega^2}{\pi^2 c^3}\left(1+\frac{a^2}{c^2\omega^2}\right)\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{2\pi\omega c/a}-1}\right]d\omega \tag{12}$$

This would give rise to a temperature dependent G_{eff} modified as:

$$G_{eff}(T) = G_{eff}\left(1-\frac{T^2}{T_{\max}^2}\right) \tag{13}$$

(Vanishing for $T = T_{\max}$, perhaps averting space-time singularities) [25]. Applications to the early universe and black hole collapse are relevant.

4 Energy levels in a gravitational field and solution of Dirac equation

As the high acceleration (at quantum scales) involve $\hbar$, we would have energy levels

$$E = \frac{n\hbar a}{c} \tag{14}$$

Recent experiments have tried to look for these quantum levels, of neutrons in the earth's gravitational field [17, 18]. These levels could also be observable in high energy collisions.

The equation (14) can also be written as $E = ka;\ (\hbar = kc)$. [19]

These energy levels are given by using the Bohr-Wilson rule:

$$2\int_0^{E/mg} dy\left[2m(E-mgy)^{1/2}\right] = n\hbar \tag{15}$$

giving the energy levels:

$$E_n = \left[\frac{9mgn^2\hbar^2}{32}\right]^{1/3} \tag{16}$$

where *n* is an integer. (Varying accelerations can also be incorporated like $g = g(y)$, etc.

A solution of Dirac equation for electron energy levels in a gravitational field gives similar results, i.e. [2, 21]

$$\left[c\alpha.p + \beta(mc^2 + mgx)\right]\Psi(x) = 0 \tag{17}$$

Using the two component form, we have:

$$\left[\left(1-\alpha\frac{\hbar g}{c}\right)p^2+\frac{g^2}{c^2}\left(x+\frac{mc^2}{g}\right)^2\right]\phi_1=\left(E^2+\frac{\hbar g}{c}\right)\phi_1 \tag{18}$$

(Similarly for ϕ_2)

This gives the relativistic energy levels as:

$$E_n^2=\left[2n\hbar gc+n^2\alpha g^2c^2\right]^{1/2} \tag{19}$$

For the case of a black hole we have:

$$a=\frac{c^4}{GM} \tag{20}$$

where M is the black hole mass. It turns out that there is a universal upper limit to Force as $F_{\max}=\frac{c^4}{G}$.

This gives the corresponding Unruh temperature as:

$$T_u=\frac{n\hbar c^3}{GMk_B} \tag{21}$$

So we have a prediction: we should get a set of energy levels apart from $n=1$ (Hawking temperature). This could imply gravitational Landau levels, $\hbar g/c$, playing role of gyrofrequency.

For primordial black holes with cut-off lifetime $t\sim 10^{18}s$, these levels would be $\sim 100MeV$ (γ rays)!

In this connection it is of great interest that a model of energy levels for black holes has been developed for large n by Corda and his collaborators. [26, 27, 28, 29]

In these papers remarkably consistent with the above equations (17-21), these authors show that black hole entropy is proportional to the quantum level of black hole. In [26], the spectrum of black holes quasinormal modes is reanalysed and the usual entropy relation is a function of quantum overtone number also. Earlier J W York [30] had associated the black hole entropy with the quasinormal modes. Sub-leading corrections to the entropy are also modified, now being directly with the quantum number n. Our equation (21), also has a similar consequence, the entropy also now depends on the quantum number as temperature is also connected with n.

Again the BH quantum spectrum gives quasinormal modes quantum overtone corrections to the BH entropy [27, 28] and the deviation from the thermal character of Hawking radiation harmonizes BH Hawking radiation with QNM and this is also generalised to non extremal Reissner-Nordström and Kerr BH's [28]. In summary, the effective energy levels of the black hole is not just the Hawking radiation but involves quantum levels depending on the quantum numbers.

In this context, we have the area operator in quantum gravity which implies a quantised area spectrum leading to discretized energy levels. Connection with super-radiance, wherein external radiation modes $\omega < m\Omega_H$, (Ω_H being the rotating black hole angular velocity) are amplified can also be included in this picture. A whole rich spectrum of energy levels can exist, apart from just the thermal Hawking radiation. Attempts to detect Hawking radiation should keep these aspects in mind.

5 Spiral trajectories of accelerated particles

The trajectories followed by accelerated particles (which lose energy by radiation and consequently have damped motion are well known to be exponential spirals. We have the equation of motion:

$$m\ddot{r} = \frac{e^2}{6\pi c^3}\dddot{r} + eV_{fluc} \tag{22}$$

Essentially, $\ddot{r} \propto \dddot{r}$. This implies a path given by, $a = e^{kt}$, where k is a constant. So the path is given by, $S = k''e^{k't}$; i.e. exponentially increasing (a logarithmic spiral).

This provides naturally an alternative picture for the inflation in the early universe. Thus we could have a primordial field accelerating the charged particles, where motion would be damped by radiation. So their path exponentially increases.

The high energy radiation could pair produce and create a shower of new particles. When the field weakens over a distance the particles stop being accelerated and motion becomes parabolic (or linear). These aspects will be quantitatively dealt with in a later paper. Again charged particles approaching the singularity in gravitational collapse could also be expected to follow such paths as their motion is damped by radiation emitted (as they are accelerated). The importance of spiral paths and the role of acceleration have been emphasized in ref. [19].

We recall the key property of the exponential spiral (logarithmic spiral) $R = R_0 e^{k\phi}$, i.e. the arc length from any point on the spiral to the centre (pole) is ***finite*** although it takes infinitely many rotations to reach the pole. This has implications for the singularity in black hole collapse. Again as Newton pointed out in his '*Principia*' if the law of gravity had been inverse cubic (this would be the case if there were an extra space dimension) an allowed orbit would be a logarithmic spiral (or exponential spiral).

Again relative acceleration between particles can be related to Riemann curvature of the underlying space time. Thus changes in acceleration would manifest as curvature changes. The differential acceleration would correspond to a temperature difference:

$$\Delta T = \frac{\hbar \Delta a}{2\pi k_B c} \tag{23}$$

The ΔT can be related to curvature change Δk. So analogous to the heat conduction equation which describes heat flow via:

$$\frac{\partial T}{\partial t} \propto k\nabla^2 T \tag{24}$$

We would have curvature flow (Ricci flow) in the dynamic space time of quantum micro-world.

T is a constant corresponds to thermal equilibrium (no heat flow), while k is a constant implies 'geometric' equilibrium of the underlying space time manifold. [20]

6 Space-time defects

Again space time defects could explain quantization of action (origin of $\hbar$) and quantum of electric charge [21]. Thus the torsion associated with a space-time defect, denotes the closure failure of a loop, in a surface element.

$$\oint Q dA = n\left(\frac{\hbar G}{c^3}\right)^{1/2} \tag{25}$$

Quantification of closure in the Planck units could give a clue to the origin of $\hbar$, which would now be topological. Oscillations of the defect could give rise to acceleration through second time derivative of torsion.

Similarly second time derivatives of the curvature would give rise to angular accelerations, i.e. $\frac{\partial^2 \Delta\theta}{\partial t^2} \propto RdA$ over a torsion would arise from spin density, i.e. would give rise to a gravitomagnetic force (or gravitoferromagnetic effect) [22, 23]. Thus:

$$Q = \frac{4\pi G\sigma}{c^3} \tag{26}$$

σ is the spin density ($N\,\hbar/r^3$ in a region of extent r, or $\sigma = N\,\hbar/r^3$).

So Q is analogous to a magnetic dipolar field, falling off as $1/r^3$ (as seen from equation (26)).

Hence $\oint QdA$ gives 'gravitomagnetic flux' over surface dA, analogous to $\hbar c/e$ for the magnetic flux. This in turn confirms the possibility of gravitational Landau levels, $\left(n+\frac{1}{2}\right)Q.S$, $\frac{Q.S}{\hbar}$ is the 'gravito-gyro frequency'!

This has implications for strong gravitational fields, near black holes which can give a repulsive force for spinning black holes. Strong repulsive forces have been involved [19] to describe ejection of particles in jets around black holes. Torsion effects (gravitomagnetic forces) could give rise to a finite radius for collapsing matter preventing a singular state, i.e. inside the horizon the minimum radius would be:

$$R \approx \left(\frac{GS^2}{c^4 m}\right)^{1/3} \tag{27}$$

S is the total spin. For a solar mass this gives $\sim 10^5 cm$. [24, 25]
For a derivation of equation (27), see [31]

7 Trapping of particles due to high acceleration

As high acceleration is locally a strong gravitational field, the usual criterion for gravitational confinement (of particles and fields) i.e. $\frac{Gm^2}{r} \geq mc^2$, can be written as:

$$\begin{aligned} r &\leq \frac{Gm}{c^2} \\ \frac{\hbar a}{c} &\geq mc^2 \\ a &\geq \frac{mc^3}{\hbar} \end{aligned} \tag{28}$$

We can get the same by substituting $\hbar c/m^2$ for G_{eff} in equation (2) and (3). So this can be written as:

$$\frac{G_{eff} m^2}{r} \geq mc^2 \tag{29}$$

TeV black holes would correspond to $G_{eff} \approx \frac{\hbar c}{m_w^2} \geq mc^2$, where m_w is the weak boson mass.

A whole range of values of G_{eff} is possible as a function of energy ($G' = 10^{14} G$). *TeV* 'black holes' were first discussed in ref. [32] in this context. See also ref. [21, 33]. Also in connection with strings see ref. [31, 34, 35].

One can also have gravitational trapping in vortices if frequency $\omega > c/\lambda^{1/2}$, λ being the vortex scale and c the velocity of light.

Typically the time scale is $\sim \frac{\lambda^2}{\omega \nu}$, ν is the kinetic viscocity. Such phenomenon could occur in relativistic fluids (e.g. Quark gluon plasma, high temperature electron-positron plasma etc.) and could be tested in future experiments. As stated above strong gravitomagnetic forces could lead to formation of jets in relativistic fluids (also in accretion discs in close vicinity of massive black holes). The field could collimate the jet over long distances. At relativistic velocities, gravitomagnetic forces could be comparable to static gravitational forces. There are interesting spin-torsion, spin-orbit and spin-spin effects in such systems (brief discussion in ref. [21] and references therein). A more quantitative study is being explored. Such forces could even prevent the merger of black holes on their close approach.

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COMPLETE LIFT OF A STRUCTURE SATISFYING $F^3 + F = 0$ TO VECTOR SUBBUNDLE

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Abstract

In this paper, we have studied complete lift of a structure satisfying $F^3 + F = 0$, F-structure, to vector subbundle $\tilde{E}$ of vector bundle (E, π, M) and obtained the integrability conditions in the vector subbundle $\tilde{E}$.

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1 Introduction

The geometry of F-structure on a differentiable manifold have been studied by [1, 5, 6, 7] and [9]. Yano and Ishihara [8, 9], Yano and Kobayashi [10] and Yano and Patterson [11] have studied prolongations of F-structure to tangent, cotangent bundles and to tangent bundles of order 2.

Özkan [2], Özkan and Esin [3] and Yıldırım and Esin [12] recently studied the vertical and complete lifts of tensor fields and connections to vector bundle.

In the present paper, we have studied the complete lifts of F-structure to vector subbundle $\tilde{E}$ of vector bundle (E, π, M) and obtained the integrability conditions in the $\tilde{E}$.

In §2, we recall some results stated in [2, 3, 12] which are indispensable in the subsequent sections.

In §3, we study the complete lift F^C of a tensor field F of type $(1,1)$ to the vector subbundle $\tilde{E}$.

In the last section, the complete lifts of F-structure to vector subbundle $\tilde{E}$ are studied and integrability conditions of a F-structure, in the vector subbundle $\tilde{E}$, are obtained. The results obtained in this section are coincides with earlier results on tangent bundles TM [9] and [10] where tangent bundle was taken instead of the vector bundle E.

2 Preliminaries

Let M be an $n-$dimensional differentiable manifold of class C^∞, (TM, π_M, M) its tangent bundle and (E, π, M) vector bundle with the base space M. Let (U, x^i) be a local coordinate system on M and $(\pi^{-1}(U), x^i, v^a)$ be induced local coordinate system on E defined by $x^i(u) = x^i(\pi(u))$ and $u = v^a(u)\rho_a$ for all $u \in \pi^{-1}(U)$, where $1 \leq i \leq n$, $1 \leq a \leq p$ and $\rho_1, ..., \rho_n$ are adapted sections. In particular for $E = TM$, we consider induced local coordinate

system $(\pi_M^{-1}(U), x^i, y^i)$ on TM as $x^i = x^i(\pi_M(u))$ and $y^i(u) = dx^i(u)$ for all $u \in \pi^{-1}(U)$. The indices h, i, j, k, l, r, s and a, b run over the range $\{1, 2, 3, ..., n\}$ and $\{1, 2, 3, ..., p\}$, respectively.

Let $\dim E = n + p$ and $n \geq p$. We denote a surjective vector bundle morphism ${}^0I : E \to TM$. Then there are the components $B_a^i(z)$ such that ${}^0I(u) = B_a^i(z)\, u^a \frac{\partial}{\partial x^i}\Big|_z$ (if $u \in E_z = \pi^{-1}(z)$, $z \in M$) for any $u \in E$. Thus ${}^0I(x^i, v^a) = (x^i, B_a^i v^a)$ and its clear that $rank B = rank\,[B_a^i] = n$ [12].

Let $\tilde{E}$ be a vector subbundle of vector bundle E such that $T\tilde{E} = Sp\{\delta_i, P_s : 1 \leq i \leq n, 1 \leq s \leq n\} \subset TE$ and $T^*\tilde{E} = Sp\{dx^i, P_s^* : 1 \leq i \leq n, 1 \leq s \leq n\} \subset T^*E$ where $\delta_i = \frac{\partial}{\partial x^i} - q_j^s \Gamma_i^j P_s$, $P_s = B_a^s \frac{\partial}{\partial v^a}$, $P_s^* = q_j^s B_a^j dv^a$, $\Gamma_i^j = \frac{\partial B_a^j}{\partial x^i} v^a$, $[q_i^s] = \left(BB^T\right)^{-1}$ and $[\tilde{q}_i^t] = BB^T$.

Recall that for the later use, definitions and properties of the vertical and complete lifts of functions and vector fields and tensor fields of type $(1, 1)$ to vector subbundle $\tilde{E}$ [2, 3, 4, 12]. We will use this fact later.

Let X and F be a vector field and a tensor field of type $(1, 1)$ in M, respectively. We denote respectively by X^V and F^V their vertical lifts to $\tilde{E}$ and X^C and F^C their complete lifts to $\tilde{E}$.

For X^i and F_j^i are the components of a vector field X and a tensor field F of type $(1, 1)$ in M, respectively, we have by definition

$$X^V = q_i^s X^i P_s,$$

$$X^C = X^i \delta_i + q_i^s \left(X^i\right)^C P_s,$$

$$F^V = \begin{bmatrix} 0 & 0 \\ F_i^s q_s^h & 0 \end{bmatrix}, \tag{1}$$

and

$$F^C = \begin{bmatrix} F_i^h & 0 \\ \left(F_i^s\right)^C q_s^h & F_t^s q_s^h \tilde{q}_i^t \end{bmatrix} \tag{2}$$

with respect to the local coordinates.

We have [2, 3] these propositions:

Proposition 2.1 *Let $F \in \Im_1^1(M)$. Then*

$$\begin{aligned} F^V X^V = 0 \quad , \quad F^V X^C = (FX)^V \quad , \\ F^C X^V = (FX)^V \quad , \quad F^C X^C = (FX)^C \end{aligned} \tag{3}$$

for any $X \in \mathfrak{F}_0^1(M)$.

Proposition 2.2

$$[X^V, Y^V] = 0 \quad , \quad [X^V, Y^C] = [X, Y]^V \quad , \quad [X, Y]^C = [X^C, Y^C] \quad (4)$$

for any $X, Y \in \mathfrak{F}_0^1(M)$.

3 Complete Lifts of Tensor Fields of Type $(1,1)$

We shall here study the complete lift F^C of a tensor field F of type $(1,1)$ in greater detail. Let I be the field of identity transformations of tangent spaces of M. Then, (1) and (2), we can easily verify that I^C is the field of identity transformations of the tangent spaces of $T\tilde{E}$, which is also denoted by I, i.e.,

$$I^C = I \quad (5)$$

and that I^V has components

$$I^V = \begin{bmatrix} 0 & 0 \\ \delta_i^s q_s^h & 0 \end{bmatrix}$$

with respect to the local coordinates in E where δ_i^s is the Kronecker delta. Thus we have

$$I^V X^V = 0 \quad , \quad I^V X^C = X^V \quad ,$$

$$I^C X^V = X^V \quad , \quad I^C X^C = X^C$$

for any $X \in \mathfrak{F}_0^1(M)$.

We have from (1) and (2)

Proposition 3.1 *A tensor field F of type $(1,1)$ is of rank r if and only if F^V is of rank r.*

Proposition 3.2 *A tensor field F of type $(1,1)$ is of rank r if and only if F^C is of rank $2r$.*

Proposition 3.3 *Let $F, G \in \Im_1^1(M)$. Then*

$$(FG)^C = F^C G^C. \tag{6}$$

Proof. *By Proposition 2.1, we have*

$$\begin{aligned} (FG)^C X^C &= ((FG)X)^C = (F(GX))^C = F^C (GX)^C \\ &= F^C \left(G^C X^C\right) = \left(F^C G^C\right) X^C \end{aligned}$$

for any $X \in \Im_0^1(M)$ and hence, by [2] Proposition 7, we have $(FG)^C = F^C G^C$, which proves Proposition 3.3. ■

As a direct corollary to Proposition 3.3, we have, taking account of (5),

Proposition 3.4 *If $P(t)$ is a polynomial in one variable t, then*

$$P\left(F^C\right) = (P(F))^C \tag{7}$$

for any $F \in \Im_1^1(M)$.

As examples, we have

$$\begin{aligned} \left(F^2 + I\right)^C &= \left(F^C\right)^2 + I, \\ \left(F^3 + F\right)^C &= \left(F^C\right)^3 + F^C \end{aligned} \tag{8}$$

for any $F \in \Im_1^1(M)$.

Let $F, G \in \Im_1^1(M)$. Then the torsion tensor $N_{F,G}$ of F and G is, by definition, a tensor field of type $(1,2)$ given by

$$\begin{aligned} 2N_{F,G}(X,Y) &= [FX, GY] + [GX, FY] + (FG + GF)[X,Y] \\ &\quad - F[GX, Y] - F[X, GY] - G[FX, Y] - G[X, FY] \end{aligned}$$

X and Y being arbitrary vector fields in M [9]. For this tensor we have

Proposition 3.5 *Let $F, G \in \Im_1^1(M)$. Then*

$$(N_{F,G})^C = N_{F^C, G^C}. \tag{9}$$

Proof. *Let $X, Y \in \Im_0^1(M)$. Then, taking account of (3), (4) and (6), we have*

$$\begin{aligned}
&2(N_{F,G})^C(X^C, Y^C) = 2(N_{F,G}(X,Y))^C \\
&\quad = ([FX,GY] + [GX,FY] + (FG+GF)[X,Y] \\
&\qquad -F[GX,Y] - F[X,GY] - G[FX,Y] - G[X,FY])^C \\
&\quad = [FX,GY]^C + [GX,FY]^C + (FG+GF)^C[X,Y]^C \\
&\qquad -F^C[GX,Y]^C - F^C[X,GY]^C - G^C[FX,Y]^C - G^C[X,FY]^C \\
&\quad = [F^CX^C, G^CY^C] + [G^CX^C, F^CY^C] + (F^CG^C + G^CF^C)[X^C, Y^C] \\
&\qquad -F^C[G^CX^C, Y^C] - F^C[X^C, G^CY^C] - G^C[F^CX^C, Y^C] - G^C[X^C, F^CY^C] \\
&\quad = 2N_{F^C,G^C}(X^C, Y^C)
\end{aligned}$$

from which we have (9), by Proposition 7 of [2]. This completes the proof. ■

Let $F \in \Im_1^1(M)$. Then the Nijenhuis tensor N_F of F is a tensor field of type $(1,2)$ defined by

$$N_F = N_{F,F} \quad ,$$

that is,

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]$$

for any $X, Y \in \Im_0^1(M)$ [9].

Proposition 3.5 implies the following:

Proposition 3.6 *For any $F \in \Im_1^1(M)$*

$$(N_F)^C = N_{F^C}. \tag{10}$$

4 Complete Lift of F-Structure

Let F be a nonzero tensor field of type $(1,1)$ and of class C^∞ on an $n-$dimensional manifold M such that [9]

$$F^3 + F = 0 \tag{11}$$

and being of constant rank r everywhere in M. Such a structure on M is called an F-structure of rank r.

Let the operators on M be defined as follows [9]:

$$l = -F^2 \quad \text{and} \quad m = F^2 + I. \tag{12}$$

From the operators defined by (12) we have [9]

$$l + m = I \quad , \quad l^2 = l \quad , \quad m^2 = m \quad , \quad lm = ml = 0 \ , \tag{13}$$

$$Fl = lF = F \quad , \quad Fm = mF = 0 \ . \tag{14}$$

For F satisfying (11), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m, respectively.

When the rank of F is r, D_l is r-dimensional and D_m is $(n-r)$-dimensional, where $\dim M = n$.

The following integrability conditions (i), (ii), (iii) and (iv) are known [1]:

(i) A necessary and sufficient condition for the distribution D_m to be integrable is that

$$N_F(mX, mY) = 0 \tag{15}$$

for any $X, Y \in \Im_0^1(M)$, N being the Nijenhuis tensor of F.

(ii) A necessary and sufficient condition for the distribution D_l to be integrable is that

$$mN_F(X, Y) = 0 \tag{16}$$

for any $X, Y \in \Im_0^1(M)$.

(iii) A necessary and sufficient condition for an F-structure F to be partially integrable is that

$$N_F(lX, lY) = 0 \tag{17}$$

for any $X, Y \in \Im_0^1(M)$.

(iv) A necessary and sufficient condition for an F-structure to be integrable is that

$$N_F(X,Y) = 0 \tag{18}$$

for any $X, Y \in \Im_0^1(M)$.

Proposition 4.1 *The complete lift F^C defines a F-structure in $\tilde{E}$ if and only if F defines a F-structure in M. Then F is of rank r if and only if F^C is of rank $2r$.*

Proof. *Let F satisfy (11) then F defines F-structure in M satisfying*

$$F^3 + F = 0$$

which in view of (7) yields

$$\left(F^C\right)^3 + F^C = 0 \quad .$$

Therefore F^C defines a F-structure in $\tilde{E}$. The converse can be proved in a similar manner.
The second part of proposition follows in view of Proposition 3.2. ■

Taking account of (6), (7), (12), (13) and (14) , we get the following formulas:

$$l^C = -(F^C)^2 \quad \text{and} \quad m^C = (F^C)^2 + I. \tag{19}$$

and

$$l^C + m^C = I, \quad (l^C)^2 = l^C, \quad (m^C)^2 = m^C, \quad l^C m^C = m^C l^C = 0 \tag{20}$$

$$F^C l^C = l^C F^C = F^C \quad , \quad F^C m^C = m^C F^C = 0 \ . \tag{21}$$

From (19), (20) and (21), the complete lifts l^C of l and $m^C$of m are complementary projection tensor in $\tilde{E}$. Thus there exist in $\tilde{E}$ two complementary distributions D_{l^C} and D_{m^C} determined by l^C and m^C, respectively. The distributions D_{l^C} and D_{m^C} are respectively the complete lifts $(D_l)^C$ and $(D_m)^C$ of D_l and D_m. Let us denote N and $\tilde{N}$ the Nijenhuis tensor of F and F^C, respectively.

Proposition 4.2 *The complete lift D_{m^C} of a distribution D_m in $\tilde{E}$ is integrable if and only if D_m is so in M.*
Proof. *By (3), (4) and (10), we get*

$$(N(mX, mY))^C = \tilde{N}\left(m^C X^C, m^C Y^C\right)$$

for any $X, Y \in \Im_0^1(M)$. Then (15) is equivalent to

$$\tilde{N}\left(m^C X^C, m^C Y^C\right) = 0 \quad .$$

Thus D_{m^C} is integrable if and only if D_m is integrable in M. ■

Proposition 4.3 *The complete lift D_{l^C} of a distribution D_l in $\tilde{E}$ is integrable if and only if D_l is so in M.*
Proof. *By (3), (4) and (10), we get*

$$(mN(X, Y))^C = m^C \tilde{N}\left(m^C X^C, m^C Y^C\right)$$

for any $X, Y \in \Im_0^1(M)$. Then (16) is equivalent to

$$m^C \tilde{N}\left(m^C X^C, m^C Y^C\right) = 0 \quad .$$

Thus D_{l^C} is integrable, if and only if D_l is integrable in M. ■

Proposition 4.4 *The complete lift F^C of F is partially integrable in $\tilde{E}$, if and only if F is so in M.*
Proof. *By (3), (4) and (10), we get*

$$(N(lX, lY))^C = \tilde{N}\left(l^C X^C, l^C Y^C\right)$$

for any $X, Y \in \Im_0^1(M)$. Then (17) is equivalent to

$$\tilde{N}\left(l^C X^C, l^C Y^C\right) = 0 \quad .$$

Thus F^C is partially integrable, if and only if F is partially integrable in M.
■

Proposition 4.5 *The complete lift F^C is integrable in $\tilde{E}$, if and only if F is integrable in M.*

Proof. *By (10), we have*

$$\tilde{N}\left(X^C, Y^C\right) = \left(N\left(X, Y\right)\right)^C .$$

Then (18) is equivalent to

$$\tilde{N}\left(X^C, Y^C\right) = 0 \quad .$$

Thus F^C is integrable, if and only if F is integrable in M. ■

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COMPREHENSIVE REVIEW ON SANTILLI'S MAGNEGAS IN HADRONIC CHEMISTRY

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Abstract

In this paper we have reviewed the novel chemical species of the magnecules synthesized by Santilli that comprises of individual atoms, radicals and ordinary molecules bonded through the magnetic attractive forces originating out of toroidal polarization of the orbitals of atomic electrons under strong magnetic fields. The main focus of this paper is to review the fabulous applications of Santill's magnegas. The novel magnecular species of hydrogen and oxygen find their place in fuel industry especially in fuel cells with the increase in its power, efficiency and total output. In this account we have also considered the flame temperature report of the new magnecular species of gases. We emphasize the importance of this new field.

Keywords: Magnecules, Magnegas and hadronic chemistry.

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1 Introduction

According to an U.S. Department of Energy (DOE) release, about 74 billion barrels of fossil oil, corresponding to about four trillions gallons of gasoline, are consumed in our planet per day in an estimated number of half a billion cars, one million trucks, one hundred thousand planes, plus industrial, agricultural, and military uses. The combustion of such an enormous amount of fossil fuel per day has caused increasingly alarming environmental problems, such as: 1) Measurable emission in our atmosphere of the largest amount of carcinogenic and other toxic substances; 2) Measurable oxygen depletion in our planet due to the fact that the sum of the oxygen emitted in the combustion exhaust and that recycled by plants from the emitted CO_2 is smaller than the oxygen used in the combustion (negative oxygen balance); 3) Measurable increase of carbon dioxide of dramatic proportions, with potentially catastrophic climatic changes; and other environmental problems, such as the production of poisonous NOx. The limitations of using hydrogen as the conventional fuel also had to be altered. Thus there arises a need of new species of hydrogen and oxygen called magnegases [1-7].

1.1 Alarming Environmental Problem caused by Fossil fuels, Hydrogen and Fuel Cells

Some of the cataclysmic climatic events facing by the mankind due to the large environmental disturbance are Global warming, ozone layer depletion, acid rain, change in the pattern Global warming,increase in the numbers of hurricanes and tornados, change in the pattern of rainfall and snow in many regions of the world, slow-down of the Gulf Stream and many more..... The extremely serious environmental problems caused by the disproportionate combustion of fossil fuels is as follows [3-7]:

1. Excessive release in our atmosphere of carbon dioxide. The combustion of fossil fuels releases in our atmosphere about 60 millions metric tons of carbon dioxide CO_2 per day that are responsible for the large environmental problem known as global warming or green house effect.

2. Excessive removal of breathable oxygen from our atmosphere. The

combustion of fossil fuels causes the permanent removal from our atmosphere of about 21 millions metric tons of breathable oxygen per day, a second, extremely serious environmental problem known as oxygen depletion.

3. The largest emission in our atmosphere of highly carcinogenic and toxic substances. The combustion of fossil fuels releases in our atmosphere about 15 millions metric tons of carcinogenic and toxic substances per day. This is serious environmental problems leads to atmospheric pollution, while in reality it refers to the primary source of the widespread increase of cancer in our societies.

It is generally believes that the combustion of natural gas alleviates these alarming environmental problems, with particular reference to a reduction of carcinogenic emissions. Such a view appears to be supported by the fact that natural gas does indeed burn cleaner than other fossil fuels. However, measurements conducted in the EPA certified laboratory Liphardt and associates of long island, New York, reproduced shown in table1. Table1 have indicated that, for the same computerized EPA routine, natural gas produces 0.380 grams/mile of hydrocarbons (another technical euphemism for carcinogenic and toxic substances) versus the gasoline production of 0.234 grams/mile. Therefore, under the same performance (same car with same load over the same distance at the same speed) natural gas combustion produces about 62% more carcinogenic substances than gasoline combustion [7].

It then follows that, again under the same conditions and performances, natural gas combustion implies CO_2 releases and oxygen depletions bigger than those causes by gasoline combustion. In fact, the same measurements conducted by Liphardt and Associates indicate that the CO_2 emission of natural gas combustion is 646.503 grams/mile while that of gasoline combustion is 458.655 grams/mile, namely, natural gas combustion implies an increase of gasoline emission of about 41%. In turn, the increase of the CO_2 emission implies a corresponding increase of the oxygen depletion that is also estimated to be of the order of 40%. Whenever facing the ever increasing cataclysmic climactic events caused by fossil fuel combustion, a rather widespread belief is that the solution already exists and it is given by hydro-

Table 1: Measurements of magneGas, natural gas, and gasoline exhaust by the EPA certified automotive laboratory Liphardt and Associates of Long Island, New York, conducted in November 2000.

Element	MagneGas (MG) g/mi	Natural Gas g/mi	Gasoline g/mi	EPA Standards g/mi
Hydro-carbons	0.026	0.380 2460% of MGE	0.234 900% of MGE	0.41
Carbon Monoxide	0.262	5.494 2096% of MGE	1.965 750% of MGE	3.40
Nitrogen Oxides	0.281	732 260% of MGE	0.247 80% of MGE	1.00
Carbon Dioxide	235	646.503 275% of MGE	458.655 195% of MGE	–
Oxygen	9 - 12 %	0.5%-0.7% 0.04% of MGE	0.5%-0.7% 0.04% of MGE	–

gen for the large scale fuel uses of the future because hydrogen is believed to be **the cleanest fuel available to mankind**. However, Hydrogen is indeed an environmentally acceptable fuel, but only when its production and use verify the following conditions[7]:

1. Hydrogen is produced via the electrolytic separation of water;

2. The electricity used for electrolysis originates from clean and renewable energy sources, such as those of hydric, solar or wind nature; and

3. The oxygen produced by the electrolytic process is freely released in the environment so that the subsequent hydrogen combustion leaves unchanged the existing oxygen content of our atmosphere.

However, the reality in the production and use of hydrogen is dramatically different than the above ideal conditions. In fact, hydrogen is today produced in its greatest percentage via reformation processes of fossil fuels such as methane CH_4, via the use of highly polluting electric power plants, and no oxygen is released in the atmosphere during production. Fuel is used for internal combustion engines or fuel cells, hydrogen removes atmospheric oxygen for its combustion and, under the same performance, hydrogen combustion causes more oxygen depletion than gasoline combustion. This is a major environmental problem that has been ignored by the hydrogen community until recently. The electric energy used for the production of hydrogen releases in our atmosphere more green gases and carcinogenic substances than gasoline combustion. When hydrogen is produced from the electrolytic separation of water, oxygen is sold, rather than released in the atmosphere. Also, clean energy sources for the production of electricity are virtually ignorable for our large scale fuel needs, and the electricity used for the production of hydrogen is essentially of fossil origin. The sole approach environmentally acceptable today is the study of the global environmental profile pertaining to fuels, thus including the methods for their production, transportation, storage and combustion. When analyzed on these global grounds, hydrogen and fuel cells result to be more polluting than gasoline because the current production and combustion of

hydrogen are indeed more polluting than the current production and combustion of gasoline. The above alarming environmental problems caused by all currently available fuels identify the following: expected solutions:

1. The most basic need of our contemporary societies is the development of new methods for the non-polluting, large scale production of electricity [6].

2. We have to develop new methods for the production of hydrogen that are not environmentally polluting and do not cause oxygen depletion.

3. In view of the already enormous and ever increasing demand for fuels, it is easy to predict that hydrogen alone cannot replace fossil fuels, thus requiring basically new, industrially synthesized fuels with non-polluting methods of production, transportation and storage, which fuels are internally rich in oxygen of non-atmospheric origin so as to replenish the oxygen depleted until now by fossil fuels and hydrogen combustion.

The Magnegas Technology was developed along these expected solutions.

2 The birth of Santilli's Magnegas Technology

The U. S. Department of Energy to invite Prof. Ruggero Maria Santilli in the late 1970's, then at Harvard University, to apply for research grant for the development of new clean energies and fuels. By the late 1970's, all energies of fuels predicted by quantum mechanics had been essentially discovered. Therefore, the above research grants were used for the construction of a generalization of quantum mechanics today known as Hadronic mechanics, a name indicating the primary applicability of the new mechanics in the interior of hadrons, as well as similar Physical characteristics (technically characterized by nonlocal, nonlinear and nonpotential interactions due to the mutual overlapping of the wave-packets of particles at short distances). These conditions are beyond any hope of treatment via quantum mechanics

due to its strictly local, linear and potential structure. Underwater electric arcs do not produce an environmentally acceptable fuel, because they create an essentially stationary plasma near the tips of the electrodes in which plasma CO is turned into CO_2 due to the presence of oxygen in the vicinity of a discharge. This implies the production of a combustible gas with up to 18% CO_2 before combustion, thus being environmentally unacceptable. Conventional underwater electric arcs are less efficient than existing methods of water separation because most of the separated H and O atoms recombine into water [7]. These are the reasons why, during the past 150 years since its discovery, the underwater arc was unable to reach an industrial maturity for the production of a gaseous fuel despite numerous attempts. Prof. Santilli's new arc process, called PlasmaArcFlow[7], has resolved all these problems and that is the reason why it has reached industrial maturity. In fact, Santilli's new process removes H, C, O and CO from the arc immediately following their formation, thus having only traces of CO_2, preventing the recombination of H and O into H_2O, and having an efficiency at least ten time that of standard submerged arcs. In this way, Prof. Santilli's new arc process implies a gas that is environmentally acceptable and cost effective, thus having the prerequisites for industrial maturity.

2.1 Plasma-arc-flow total recyclers

The complete elimination of unwanted liquid wastes, such as automotive antifreeze and oil waste, cooking oil waste, industrial, agricultural and marine liquid wastes, etc., as well as for the processing of crude oil into the clean burning magnegas. Figure 1 represent a completely automatic, computer controlled small Total Recycler with 50 kW AC-DC converter for the recycling of water-base liquid wastes built by Prof. Santilli in December 2000. A submerged electric arc between carbonaceous electrodes decomposes the molecules of the liquid feedstock into their atoms, forms a plasma at about 10,000 degrees F of mostly ionized H, C, O and other atoms. Continuously recirculating the liquid through the arc, PlasmaArcFlow Recyclers [7] completely eliminate the liquid waste and transform it into:

1. Clean burning magnegas that bubbles to the surface where it is col-

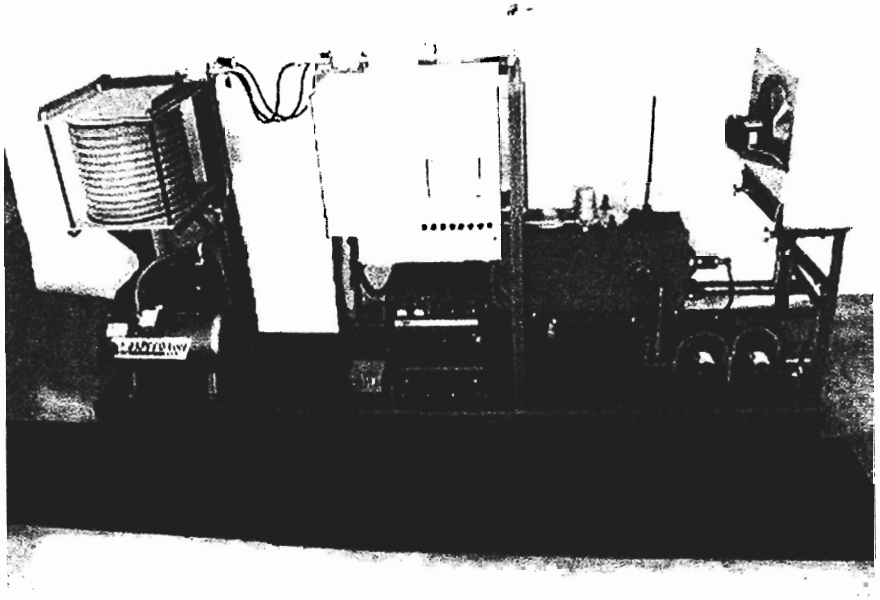


Figure 1: A completely automatic, computer controlled small Total Recycler with 50 kW AC-DC converter for the recycling of water-base liquid wastes built by Prof. Santilli in December 2000.

lected and then subjected to various purification processes;

2. Carbonaceous solids that precipitate at the bottom of the recyclers where they are periodically removed and used for the production of electrodesor as sythetic coal; and

3. Large amount of heat originating from the highly esoenergetic reactions for the formation of magnegas (such as the formation of CO that releases essentially the same heat as that from burning coal). This heat is acquired by the liquid feedstock and can be used via a heat exchanger for heating up buildings, partial self-generation of DC electricity of the arc via a steam turbine, desalting seawater via evaporation, and other applications.

2.2 Plasma-arc-flow linear recyclers

The primary purpose of Linear Recyclers is that of sterilizing bio-contaminanted waters such as city, agricultural or marine sewage(10% bio-contaminants) in an environmentally friendly way. Linear Recyclers[7] expose bio-contaminated liquids to the 10,000 degrees F of the electric arc, its very high DC currents, and its very intense ultraviolet light. All these factors eliminate any bacteriological activity while producing magnegas. Also, while passing though the electric arc, substances in suspension are turned into a carbonaceous form, while substances in solution as well as the temperature of the liquid remain essentially unaffected due to the speed of the flow. After passing through the electric arc, the liquid feedstock is passed through a centrifuge or other means for the removal of solids processed into said carbonaceous form, and finally through a filtration system. In this way, linear Recycler Recycler can process water with up to 10% bio-contaminants by producing:

1. Clean burning magnegas as for the Total Recyclers;

2. Carbonaceous materials that are used for the production of electrodes or for use as synthetic coal as for the Total Recyclers; and

3. Filtered water that is as transparent as tap water, yet contains the original substances in solution, thus being excellent for irrigation.

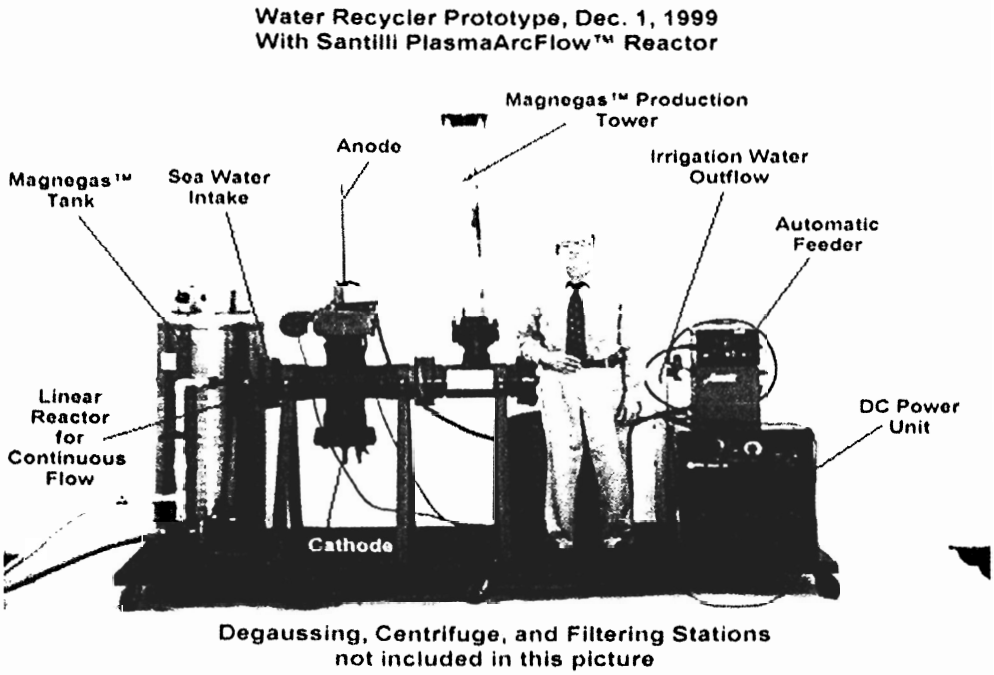


Figure 2: A prototype of the Linear Recycler with 50 kW built by Prof. Santilli in October 2000.

A prototype of the Linear Recycler with 50 kW built by Prof. Santilli in October 2000 shown in figure 2. Linear Recyclers too are environmentally very friendly because they release no contaminants of any type, cause no noise pollution, and produce no odor. As such, they are preferable over existing methods of sewage treatment, besides being cost competitive and producing a clean burning gaseous fuel.

2.3 Plasma-arc-flow gas and specialty recyclers

These recyclers are conceived and constructed for special purposes, such as:

1. The processing of gaseous (rather than liquid) feedstocks for turning conventional hydrogen, oxygen and other gases into the novel Magne-Hydrogen, MagneOxygen and other new species ;

2. The conversion of existing fossil fuels into a form without hydrocarbon content, thus being dramatically cleaners than their conventional form;

3. New processes of hydrogenization;

3 Magnegas

The electromagnecules are clusters generally composed of individual atoms, parts of conventional molecules (including radicals) and normal conventional molecules under a new internal bond originating from the electric and magnetic polarizations of the orbits of at least some peripheral atomic electrons [1,7-15]. Due to dominance of magnetic over electric polarizations as described by figure 3. Electromagnecules are generally called Santilli magnecules[16-22].

The basic concept involved in formation of magnecules is that when an atom is exposed to sufficently strong external magnetic field acquire a toroidal distribution of electron charge with consequential creation of a magnetic dipole N-S caused by rotation of electron charges by toroidal polarization. Such a dipole gets aligned along the symmetry axes of the toroidal distribution so as to possess the magnetic properties opposite to

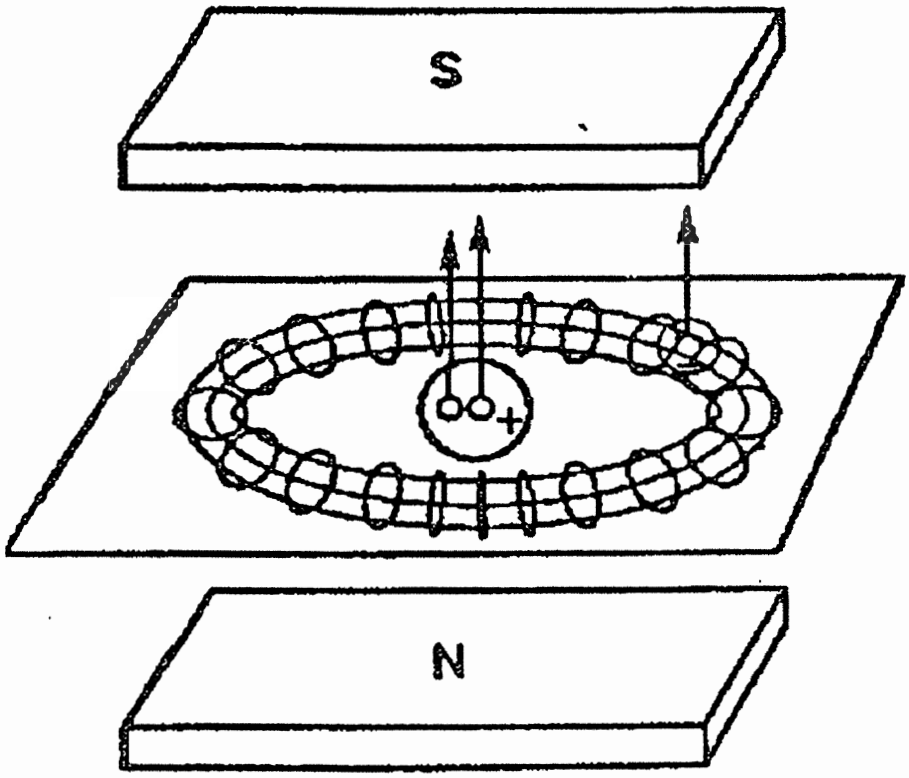


Figure 3: A schematic view of the toroidal polarization of the orbits of peripheral atomic electrons when subjected to a sufficiently strong external magnetic field. Note the additional polarization of the intrinsic magnetic moments of nuclei and of peripheral atomic electron, thus resulting in the availability of three parallel magnetic moments all suitable to create a new bond.

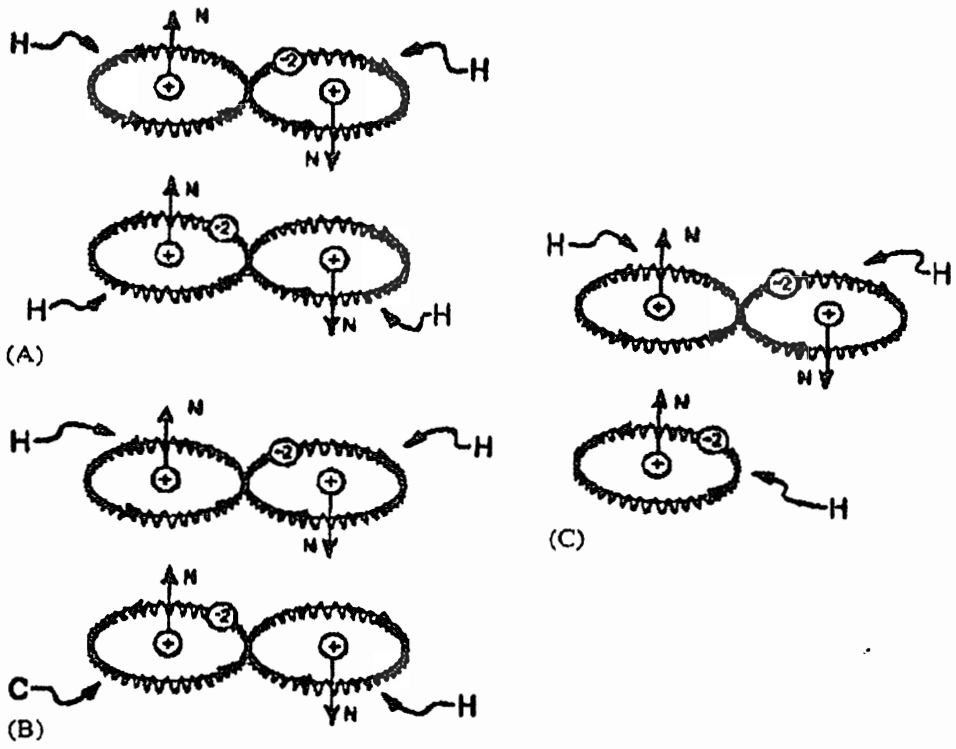


Figure 4: A schematic view of elementary magnecules composed of a diatomic molecule bonded to an individual atom or to another diatomic molecule via opposing magnetic polarities of the toroidal polarization of the orbit of the orbit of individual atoms.

the external ones. Atoms, radicals or molecules with toroidal polarization of their atomic orbits then bond to each other in chains of opposing polarities N-S-N-S- resulting in the formation of magnecules shown in figure 4. A schematic view of elementary magnecules composed of a di-atomic molecule bonded to an individual atom or to another diatomic molecule via opposing magnetic polarities of the toroidal polarization of the orbit of the orbit of individual atoms is depicted by figure 4. Simple calculations show that such a field is strong and 1415 times stronger than the intrinsic magnetic field of the nucleus of hydrogen. Thus toroidal polarization of orbits of peripheral atomic electrons creates a new field sufficiently strong to originate a new chemical species. An important feature of magnecules is that the magnetic polarization occurs in each individual atom rather than a molecule as a whole. This implies that the new chemical species of magnecules can be formed for all possible gases irrespective of whether they are diamagnetic or paramagnetic. Thus the primary technological objective for the industrial production of substances with magnecular structure depends on the control of the space distributions of the orbits of individual atoms, rather than molecules. The new chemical species of magnecules most investigated till date is Santilli magnegas produced by recyclers called hadronic reactors of molecular type. This particular gas is produced by flowing a liquid feedstock (such as fresh water or salt water, antifreeze and oil waste, city and farm sewage, crude oil, etc.)through a submerged electric arc between carbon base consumable electrodes[2,7,14,24].

3.1 Magnehydrogen

The new species is generically denoted MH and its individual clusters are denoted MH_n, n = 2, 3, 4, . to specify the number of hydrogen atoms per cluster. Most of the environmental pollution is caused by fossil fuel is due to chunks (diatomic radicals) of uncombusted fuel that may be carcinogenic primarily because consisting of incomplete combustion of fuel. Therefore, Santilli has proposed and synthesized a new chemical species called as magnecules under strong magnetic field [1,4,5,10]. Santilli's magnecules are in gaseous, liquid and solid states consist of stable clusters composed of conventional molecule or diatomic radical with individuals atoms bonded together

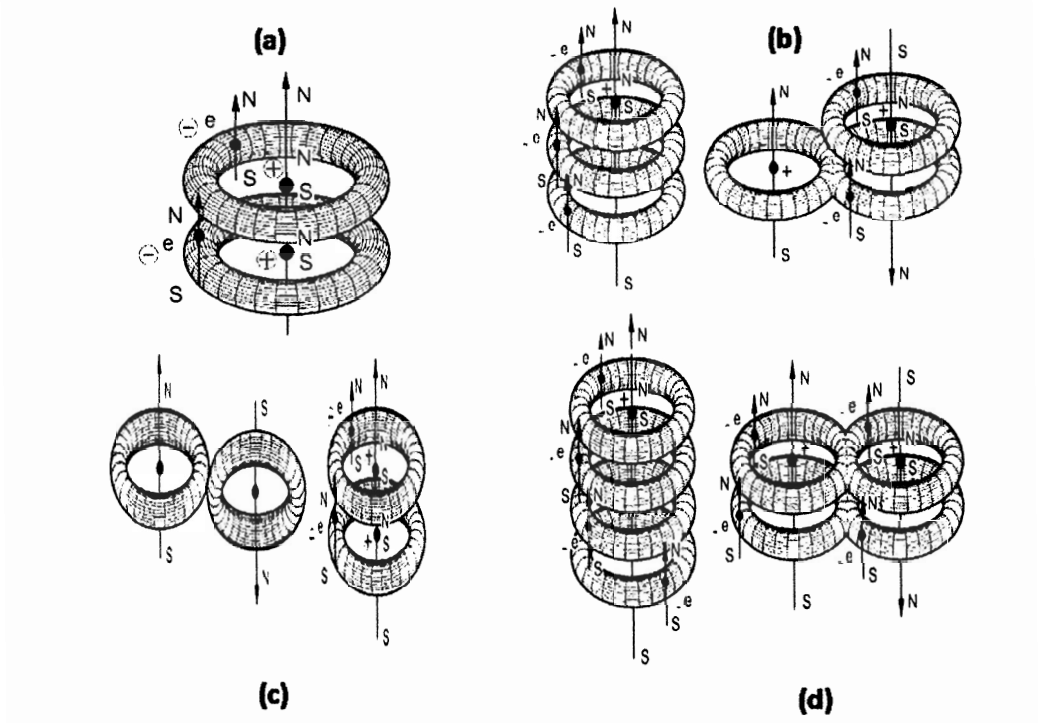


Figure 5: Conceptual rendering of Santilli's magnecules. (a) Magnehydgrogen (MH), two hydrogen atoms forming a magnecule $MH_2(H\times H)$, (b) A cluster MH_2 in MH composed of H-H and H×H molecules, (c) A cluster MH_3 in MH composed of the magnecular species H-H×H and H×H×H, (d) A cluster of MH_4 in MH composed of the magnecular species H×H×H×H and H-H×H-H (the species is H-H×H×H has not been shown)

by opposing magnetic polarities of toroidal polarization of the orbit of at least the peripheral atomic electrons when exposed to sufficiently strong external magnetic fields, as well as the polarization of the intrinsic magnetic moment of nuclei and electrons [1,5-7,10-23]. A population of magnecules constitutes a chemical species when essentially pure i.e. when molecules or other species are contained in very small percentages in a directly identifiable form. To our best understanding, the most plausible interpretation of the new species of MH is that originally presented by Santilli in Ref. [14], namely, a multiple of the specific weight under a high Hydrogen percentage is evidence of a new clustering of H-atoms which cannot possibly be of valence type due to the evident absence of the valence electrons necessary for a quantitative representation of the clustering of many different atoms. In essence, the new chemical species of magnecules can be defined as clusters of individual atoms (H, O, C, etc.), diatomic free radicals (HO, CH, etc.) and ordinary molecules (H_2, CO, H_2O, etc.) bonded together by attractive forces between opposing magnetic polarities of toroidal polarizations of atomic orbitals, as well as the polarization of the magnetic moments of nuclei and electrons (a conceptual rendering of MH_n in Figure 5). Santilli first developed, the so-called PlasmaArcFlowTM Reactor, that converts various liquids into a combustible gaseous fuel known as MagneGas (MG). The gasification is achieved via a submerged DC electric arc between carbon electrodes that, under sufficient powers (of the order of 300 kW or more) is capable of producing at atomic distances the high values of the magnetic field necessary for the polarization of electron orbitals into toroids (estimated as being of the order of 1012 Gauss).

3.2 Synthesizing clean new fuels via submerged electric arcs

It is well known that our planet does not possess a fuel in its natural resources capable of resolving the indicated environmental, technological, and financial problems. Therefore, the fuel of the future must be synthesized. synthesized new fuels must meet the following requrements;

1. New fuels must avoid the alarming environmental problems, namely, have no carcinogenic substances in the combustion exhaust, be inter-

nally rich in oxygen so as to avoid the oxygen depletion, and eliminate or reduce the emission of carbon dioxide.

2. New fuels must be usable in existing internal combustion engines since they are the source of existing environmental problems. To be more specific in this important point, suppose that fuel cells powered by environmentally acceptable production of hydrogen reach maturity to power automobiles. This would leave completely unaffected the current exponential increase of environmental problems because existing cars will continued to be used.

3. New fuels must be friends of fossil fuels.This can be achieved in a variety of ways, such as the selection of new non-fossil fuels that can be mixed with existing fossil fuels to improve combustion exhaust, or new clean fuels could be produced from crude oil via new processes, etc.

4. New, environmentally acceptable fuels must be cost competitive with respect to fossil fuels because, no matter how good their environmental quality, their practical use will be severely limited by non-competitive costs.

5. New acceptable fuels must be safer than gasoline, such as avoid the spilling with consequential explosion that are inherent in gasoline, or avoid cryogenic liquefaction with consequential dangerous changes of state.

Professor Santilli had conducted in depth research over decades for the selection of a process that permits the industrial synthesis of new fuels meeting the above requirements. The conclusion of the research is that submerged electric arcs are much preferable over other available technologies for a variety of reasons, such as:

1. Depending on the used liquid, submerged electric arc generally produce combustible gases without hydrocarbons, thus setting the grounds for the production of new fuels without carcinogenic substances in the combustion exhaust.

2. Also depending on the used liquid, submerged electric arcs can produce new clean burning fuels that are internally rich in oxygen, thus permitting a reduction of the oxygen depletion caused by fossil fuels.

3. Submerged electric arcs are much more efficient than electrolysis in the separation of the liquid into its atomic constituents, thus providing grounds for cost competitive fuels, as we shall see better later on in this site.

The same decades of research have also shown that the technology of submerged electric arcs is deceptively simple, in the sense that it gives the erroneous perception of an easy realization. In reality, the achievement of industrial maturity for the production of environmentally acceptable new fuels produced via submerged electric arcs, and the achievement of cost competitiveness are extremely difficult since the technology implies the use of the virtual entire scientific knowledge, including particle physics, nuclear physics, superconductivity, chemistry, metallurgy, and other disciplines.

3.3 Chemical structure of magnegas

Combustion exhausts Prior to Prof. Santilli's research, it was popularly believed that the gas produced by an underwater arc is composed of 50% H_2 and 50% CO, with traces of other molecules such as CO_2 and H_2O. This belief has been disproved by numerous experimental evidence, such as[7]:

1. CO burns by producing CO_2, a gas with 50% CO should produce about 40% CO_2 in the exhaust, while the exhaust of magnegas produced from water contains only about 4% of CO_2;

2. According to well known chemical reactions, the creation of 50% H_2 and 50% CO should produce 2,250 BTU per scf of magnegas produced, while only 250 BTU/scf have been measured in magnegas produced from water, thus confirming that at most 5% CO may be contained in magnegas;

3. A gas composed of 50% H_2 and 50% CO should exhibit under mass spectroscopy two corresponding large peaks.

In reality, Gas Chromatographic Mass Spectrometers (GC-MS) analyses have indeed detected a large peak at 2 atomic mass units (amu) representing H_2, although in a percentage much less than 50%, while detecting no peak at all representing CO. The above (and other) experimental evidence establishes beyond credible doubt that magnegas does not possess a conventional molecular structure. In fact, the most important novelty of the technology is that magnegas is composed of the new chemical species of Santilli magnecules, that constitutes the only new chemical species discovered by mankind since the identification of molecules by Avogadro, Canizzaro et al. over 150 years ago.

3.4 Energy content of santilli magnegases

Most important measurement conducted on fossil fuel is that of their energy content, e.g., in British Thermal Units (BTU) per unit of volume, e.g., one America Gallon (g) or one standard cubic foot (scf). For instance, gasoline contains about 115,000 BTU/g, natural gas contains about 1,050 BTU/scf, acetylene contains about 2,300 BTU/scf, etc. The achievement of these numerical values is permitted by the fact that the release of the energy content in fossil fuels is essentially the same for all possible combustions. For instance, the energy content of natural gas is the same whether it is used as fuel for internal combustion engines, or for burning in atmosphere for home heating or cooking. Energy content of magnegas from water burning in open air[7]: about 400 BTU/scf; Energy content of magnegas from antifreeze waste burning in open air: about 600 BTU/scf; Energy content of magnegas from oil waste or crude oil burning in open air: about 800 BTU/scf.

3.5 Hypothesis of the new chemical species of magnegas

As explained earlier the hydrogen molecule is diamagnetic and so cannot acquire a total net magnetic polarity. But the orbits of the individual H atoms can aquire a toroidal polarization under a sufficiently strong external magnetic field [7].The important aspect for the hypothesis of MagneH and MagneO is that toroidal polarization of the orbits of the individual H atom

along with the polarization of the intrinsic magnetic moments of nuclei and electrons in the hydrogen molecule is sufficient for the creation of the desired new chemical species with higher specific weight as new bonds can occur between the pair of individual H atoms [1,4-5]. The creation of MagneO is comparatively simple since oxygen molecule is paramagnetic thus having free electrons to acquire an overall magnetic polarity which is absent in the case of of MagneH. Thus to realize the corresponding magnecules it is necessary to provide suitable physical conditions and geometries for the joint polarization of the atoms that would favour their coupling into chains of opposing magnetic polarities. Extensive studies of the above stated aspects have established that the primary attractive force responsible for the electromagnecules is expected to be due to the magnetic polarization of the orbitals of valence and other electrons, from space to toroidal distributions. Conventional quantum electrodynamics establishes the existence of such a polarization whenever atoms and molecules are exposed to the intense magnetic field, because it exists in the vicinity of the electric arc. Such a magnetic polarization creates magnetic North-South polarities along the symmetry axis of the toroid, which permit the stacking of atoms and molecules one after the other.

3.6 Experimental evidence of MagneH

It is important to review the experimental evidence supporting the existence of magnehydrogen. Professor Santilli has brought into limelight the problems and their solutions in the unambiguous experimental detection of magnecules. Its brief account is given below. The magnegas which was produced by using an antifreeze waste as liquid feedstock was passed through 5 Angstrom zeolite filters that essentially consisted of a microporous molecular sieve selecting a gas via molecular size exclusion. The filtered gas was then subjected to the following 3 measurements[7]:

1. The filtered gas was first subjected to analytic measurements by a laboratory via gas chromatography (GC) and other tests confirmed by FTIS. All measurements were normalized, air contamination was removed, and the lower detection limit was identified as being 0.01%. These GC-MS results are infer that magnehydrogen is composed of

99.2% hydrogen and 0.78% methane, while no carbon monoxide was detected.

2. The average specific weight of this type of MH was measured by two independent laboratories as being 15.06 a.m.u., while conventional pure hydrogen has the specific weight of 2.016 a.m.u., which meant a drastic 7.47 fold increase of the specific weight of conventional hydrogen.

3. The same type of MH that was subjected to the earlier tests was submitted to GC-MS scans via the use of a HP GC 5890 and a HP MS 5972 with operating conditions exclusively set for the detection of magnecules.

The GC-MS Scan of a MH Sample results of the third tests are in accordance with the results of GC-FTIS, the GC -MS measurements were expected to show only 2 peaks one for hydrogen at about 2 a.m.u. and that for methane at about 16 a.m.u. but the results confirmed the existence of of a large peak at about 2 a.m.u. for hydrogen ,but no presence of peak at 16 a.m.u for methane rather exhibited the presence of a considerable number of additional peaks in considerable percentages accounting to 18 a.m.u. This GC-MS scan established the existence of a magnecular structure for the concerned type of MH studied. Thus the above mentioned three tests confirm the possibility (and capability) of hydrogen production in multiples of their standard specific weights and consequential increased energy content. Thus to summarize the the above content with regard to the experimental observations appear to confirm the hypothesis of new forms of hydrogen with magnecular structure capable of producing an industrially significant increase in voltage, power and efficiency of fuel cells. Professor R. M. Santilli presented [13,15] theoretical and experimental evidence on the existence of a new species of Hydrogen he called Magnehydrogen, which consists of 99% Hydrogen, yet its specific weight (or, equivalently, molecular weight) is 7.47 times that of conventional Hydrogen.

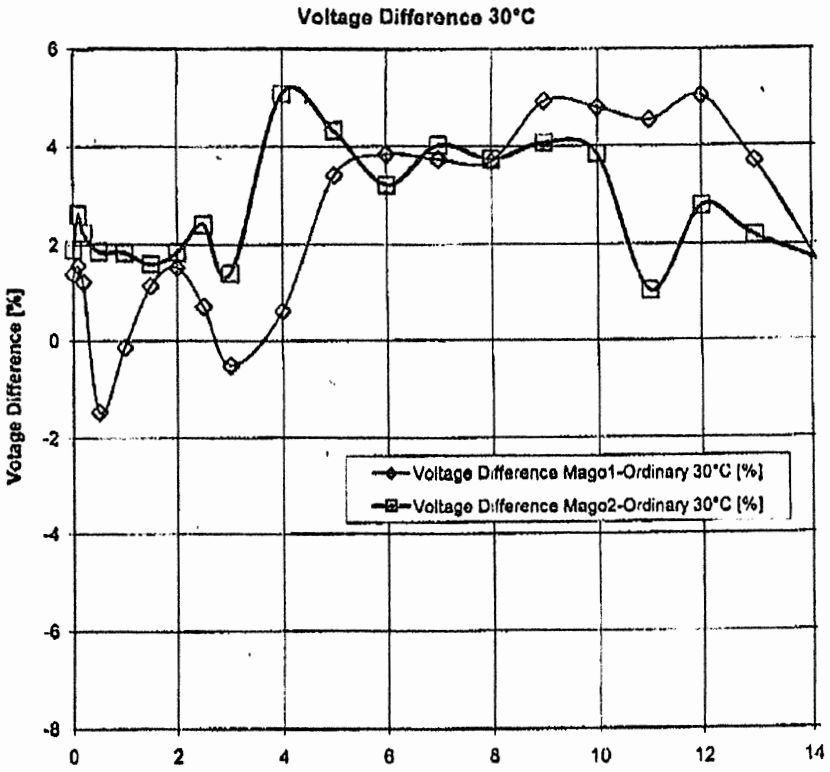


Figure 6: A schematic view of the voltage increase in a test fuel-cell operated with ordinary pure hydrogen and the two samples of magneO produced by rudimentary equipment.

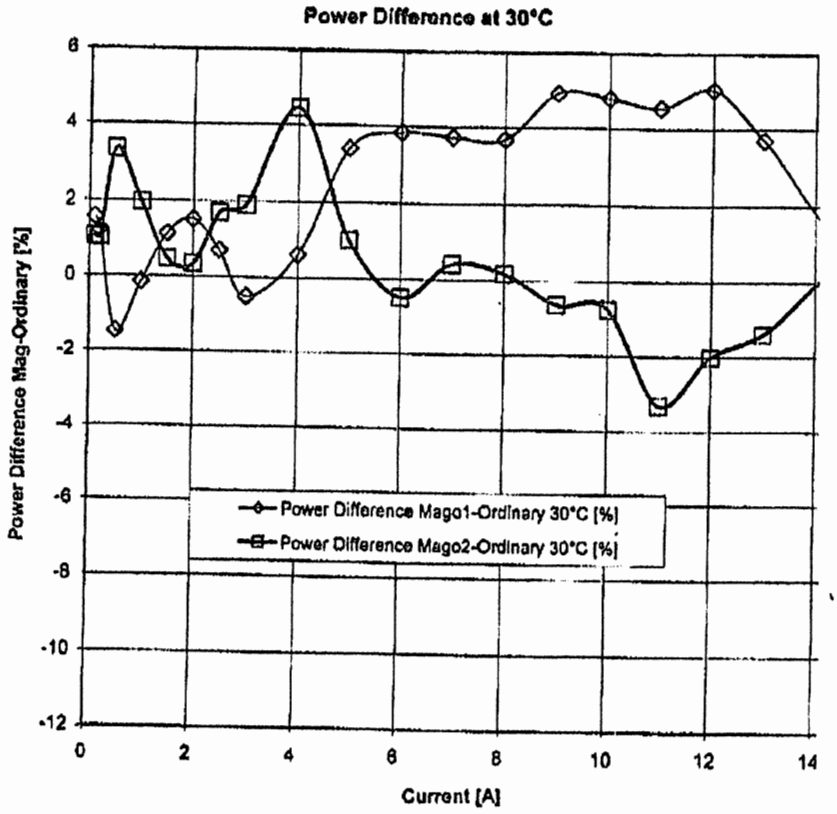


Figure 7: A schematic view of the power increase in a test fuel cell operated as in figure 6 confirming the results of the latter.

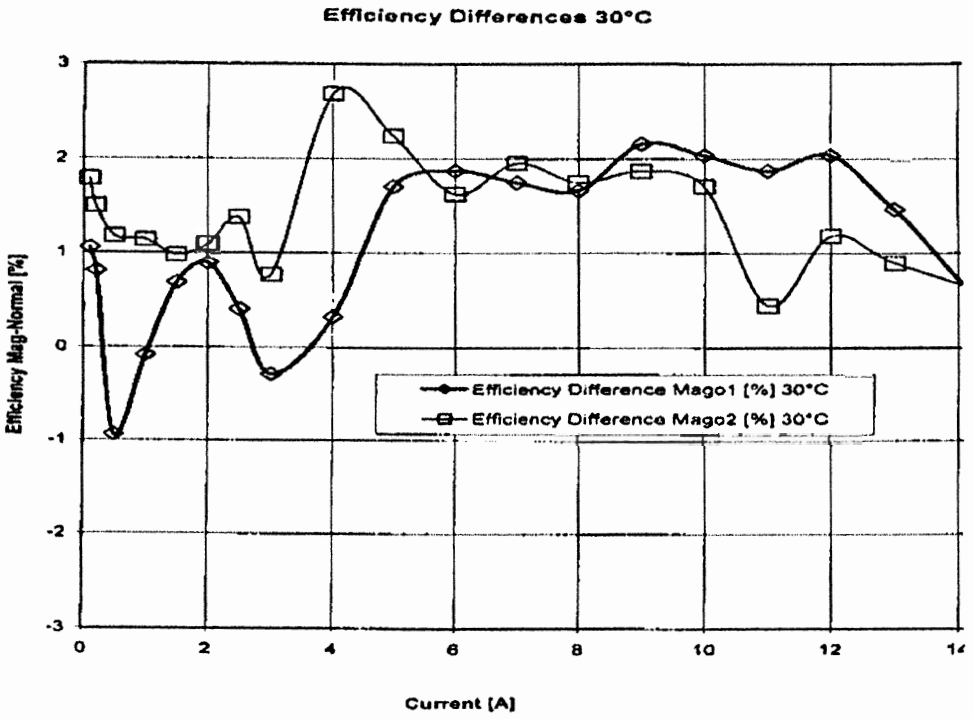


Figure 8: A schematic view of the efficiency increase in a test fuel-cell operated with ordinary pure hydrogen and the two samples of magneO.

4 Experimental evidence of magneoxygen

We present the experimental evidence of existence of MagneO. It was for this purpose that Santilli with the help of technicians constructed a rudimentary apparatus that uses the automotive sparks powered by an ordinary car battery, the system operating at about 15 psi. Two types of MagneO were produced and compared [1,4-7]. These types of MagneO were tested in place of ordinary oxygen in a 2-cell proton exchange membrane fuel-cell operated with conventional high purity hydrogen. The membrane material was Nafion 112, the catalyst in the electrodes was platinum coated on carbon; the plates for heat transfer were two nickel/gold plated material; the temperature of the fuel-cell was kept constant via conventional cooling means; the current measured via a HP 6050A electronic load with a 600 W load module; a flow rate for oxygen and hydrogen was maintained for every measurement of current; both oxygen and hydrogen were humidified before their entry into the cell; the measurements(Figures6-8) reported herein were conducted at 573K. Figures 6,7 and 8 shown a schematic view of the voltage, power and efficency increase in a test fuel-cell operated with ordinary pure hydrogen and the two samples of magneO produced. The measurements show an increase of the voltage, power and efficiency of the order of 5% as compared to the measurements when the cell was operated with MagneO1 and MagneO2. The increase was consistent for both samples except differences within statistical/experimental errors. For the appraisal of these results it is necessary to note the experimental conditions as specified above. By comparison we get a vivid idea of the industrial production of MagneO that can be produced by employing an array of arcs each operated with continuous currents of thousands of psi. It is evident that the later conditions are expected to imply a significant increase of the performance of the fuel cells when operated with MagneO.

4.1 Composition of Magnegas

Magnegas is formed under the extremely intense magnetic fields at atomic distances from the electric arc, its chemical structure is that of all possible magnecules with increasing atomic mass that can be formed from the H,

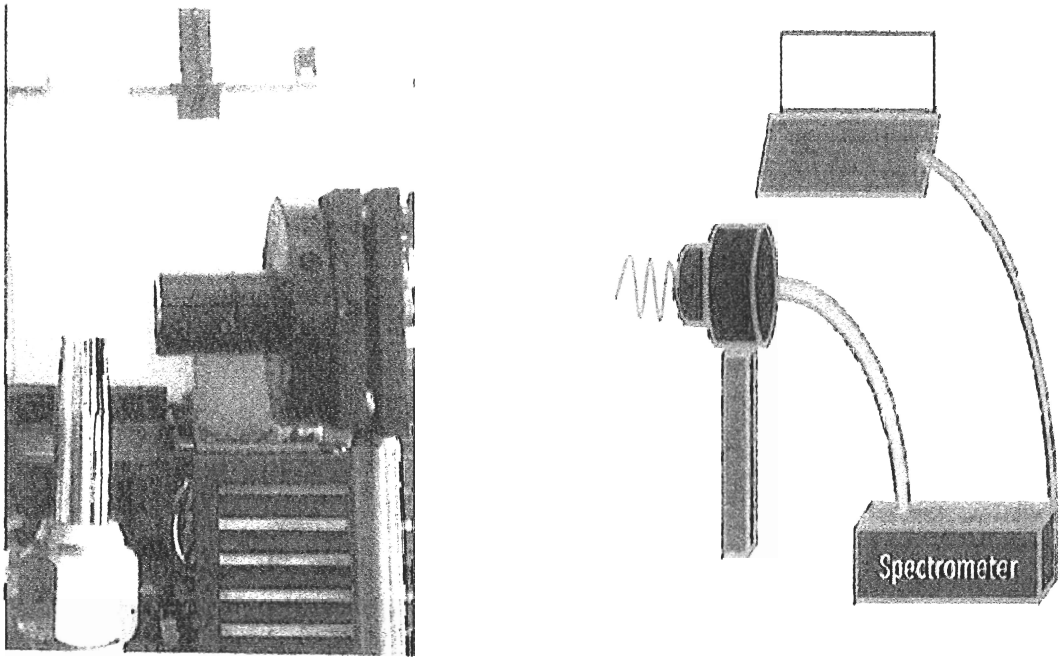
C and O atoms, such as H-H, H × H, (H-H) ×H, H×H × H; C-H, C× H, (C-H)×H, C-O, C ×O, C×(O×O), etc. where we should note that the bond between C and O can be single, double, and triple valence type as well as of magnecular nature. When reduced to atomic (rather than molecular) percentages, magnegas produced from pure water is composed of about 66%H, 22%O, 11%C. The combustion exhaust of magnegas are given by 50-55% H_2O, 12 -14% O_2, 5 -7% CO_2, atmospheric gases; where the CO_2 content originates from the combustion of conventional carbon monoxide, that of triple bonded nature O and the stochiometric ratio magnegas/atmosphere is taken into consideration.

4.2 Production of Magnegas

In PlasmaArcFlow Reactor consist with a submerged DC electric arc between carbon base electrodes to achieve the complete recycled a liquid waste [7]. The arc decomposes the molecules of the liquid into its atomic constituents; ionizes the same; and creates a plasma of mostly ionized H, C and O atoms at about 3500 K. The flow of the liquid through the arc then continuously removes the plasma from the arc following its formation [7-14]. Said plasma then cools down in the surrounding liquid, and a number of chemical reactions take place resulting in the formation of magnegas which bubbles to the surface of the liquid where it is collected for industrial or consumer use. Magnecules were directly detectable in the GC- or LC-MS.

5 Magnegas Flame Spetra

Dr. Robert Alfano, distinguished professor form the city college of New-York had performed the experiment using emission to measure the spectra from flames of Magnegas to determine the emission lines and temperature of Magnegas [25]. In this portion of the paper it is briefly described about the confirmation of the energy efficiency of the magnegases that has been supported by the experiments performed by Magnegas Corporation using emissions to measure the spectra from flames of Magnegas to determine the emission lines and temperature of Magnegas.



a b

Figure 9: a) Setup of lens in front of magnegas flame b) Experimental setup.

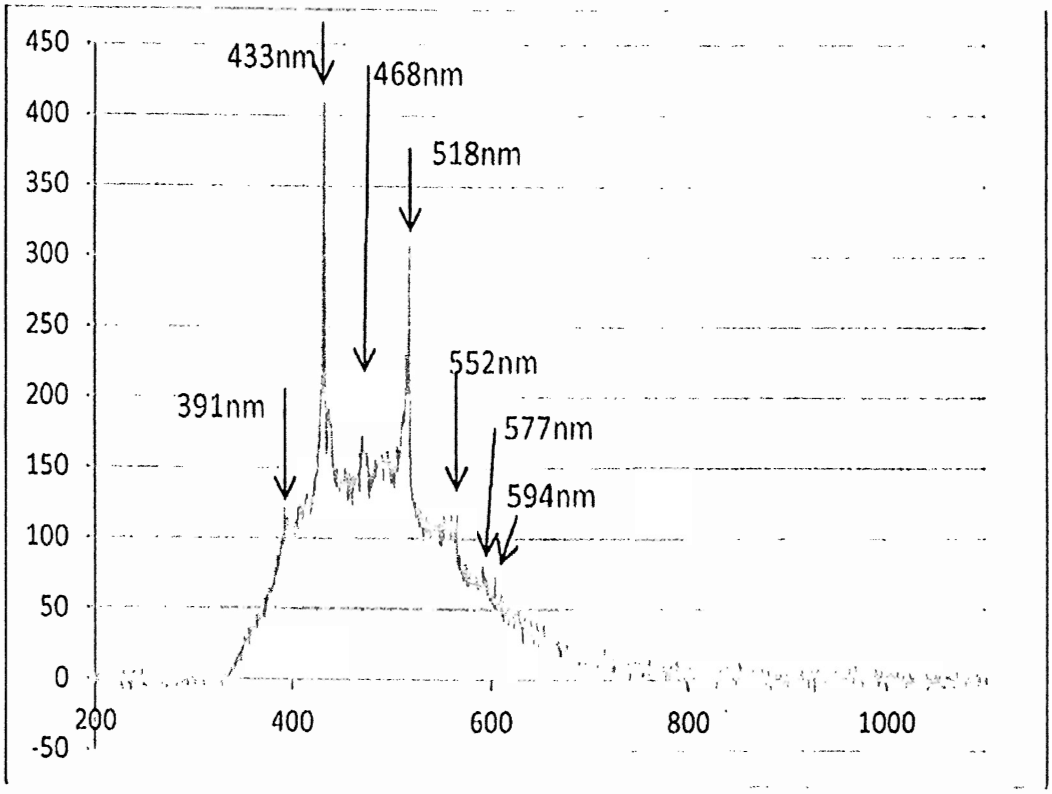


Figure 10: Magnegas from crude oil, flame only,3 mm from the base. Emission lines are indicated with arrows.

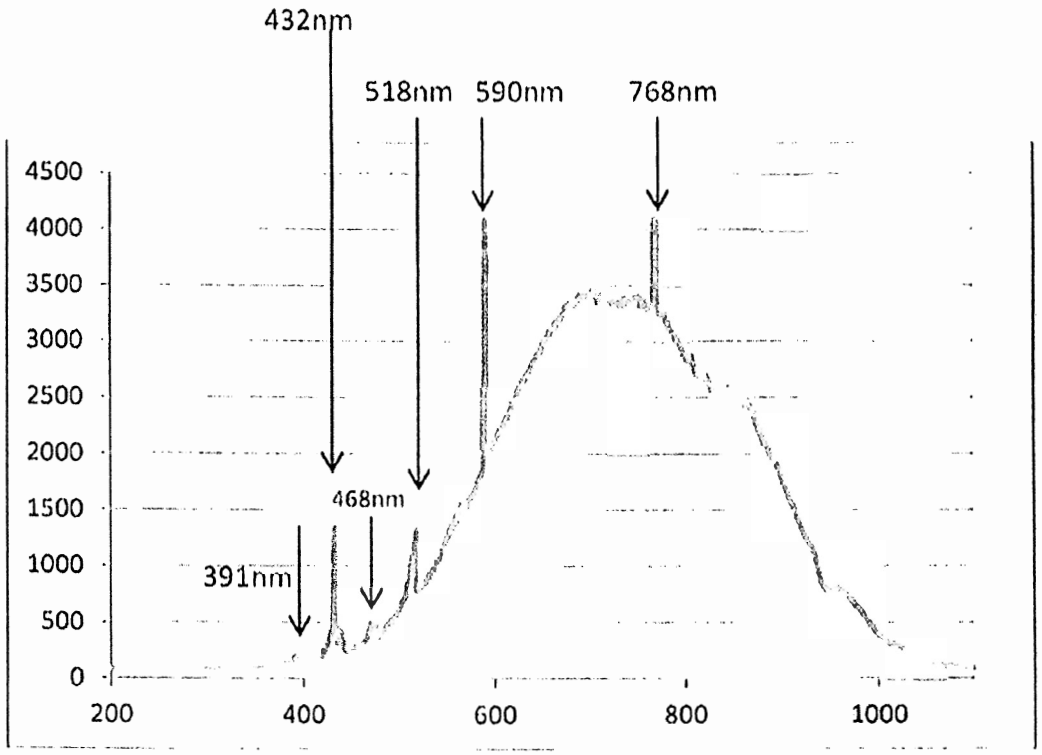


Figure 11: Flame of magnegas from crude oil at 3 mm above the surface of metal while cutting it approximately 2 mm from the base of flame

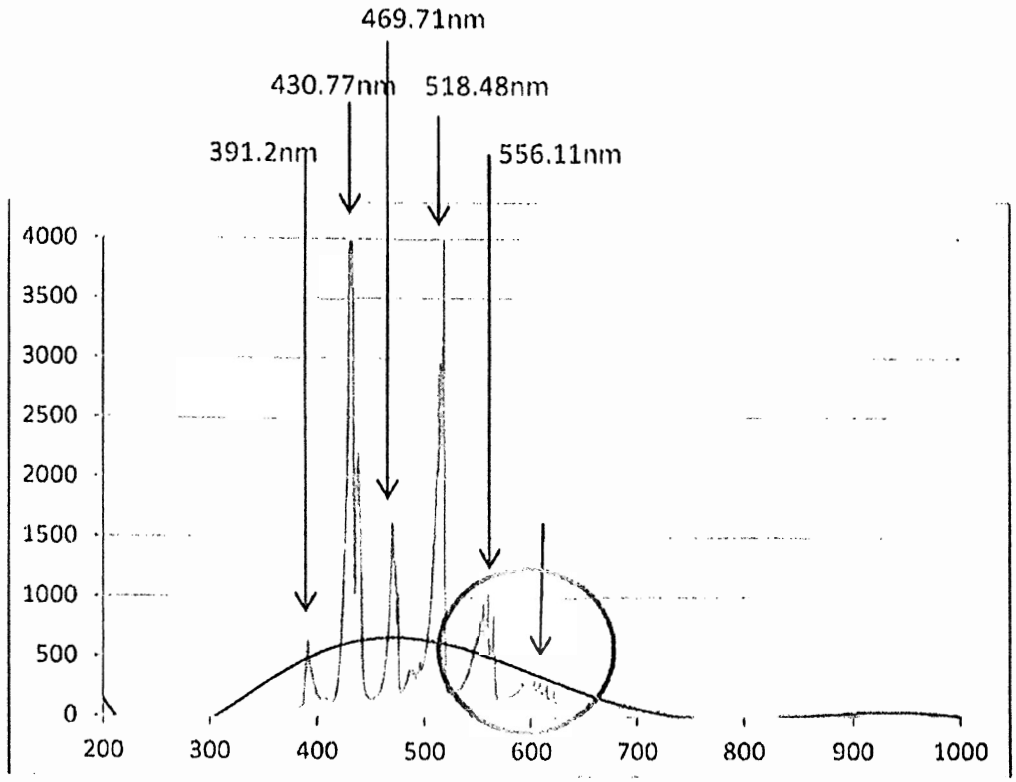


Figure 12: Magnegas from antifreeze flame only,3 mm from the base. Emission lines are indicated with arrows.

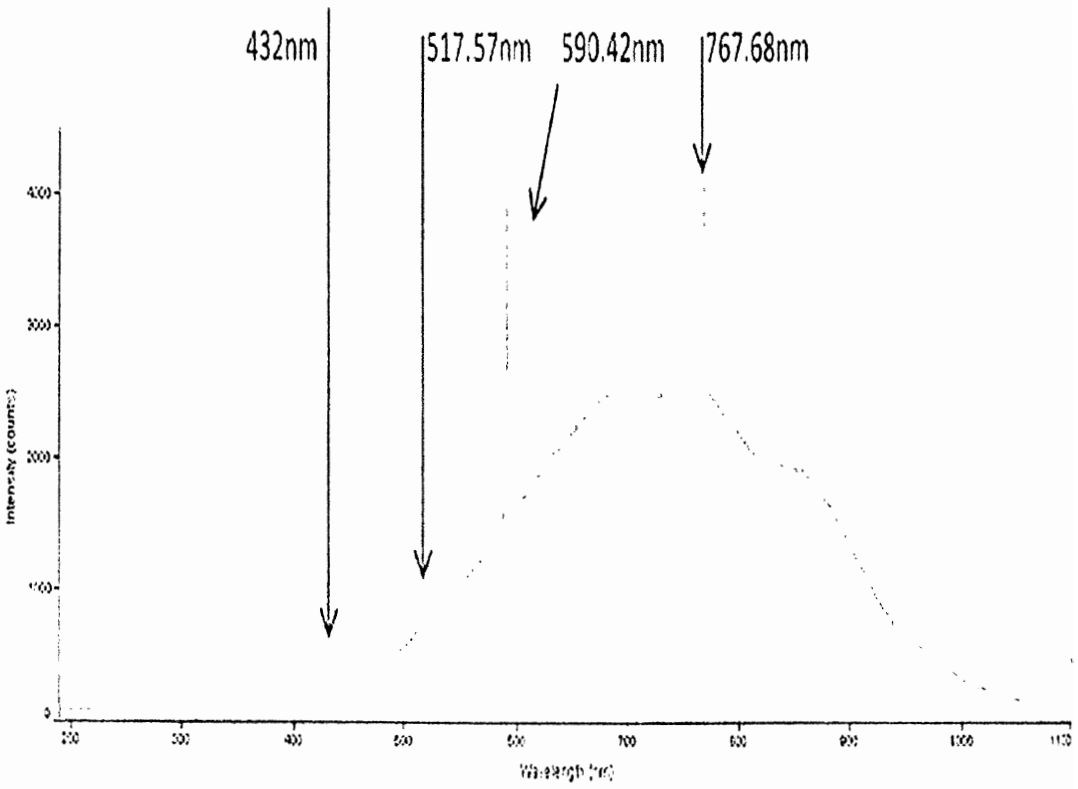


Figure 13: Flame of magnegas from antifreeze at 3 mm above the surface of metal while cutting it (approximately 2 mm from the base of flame

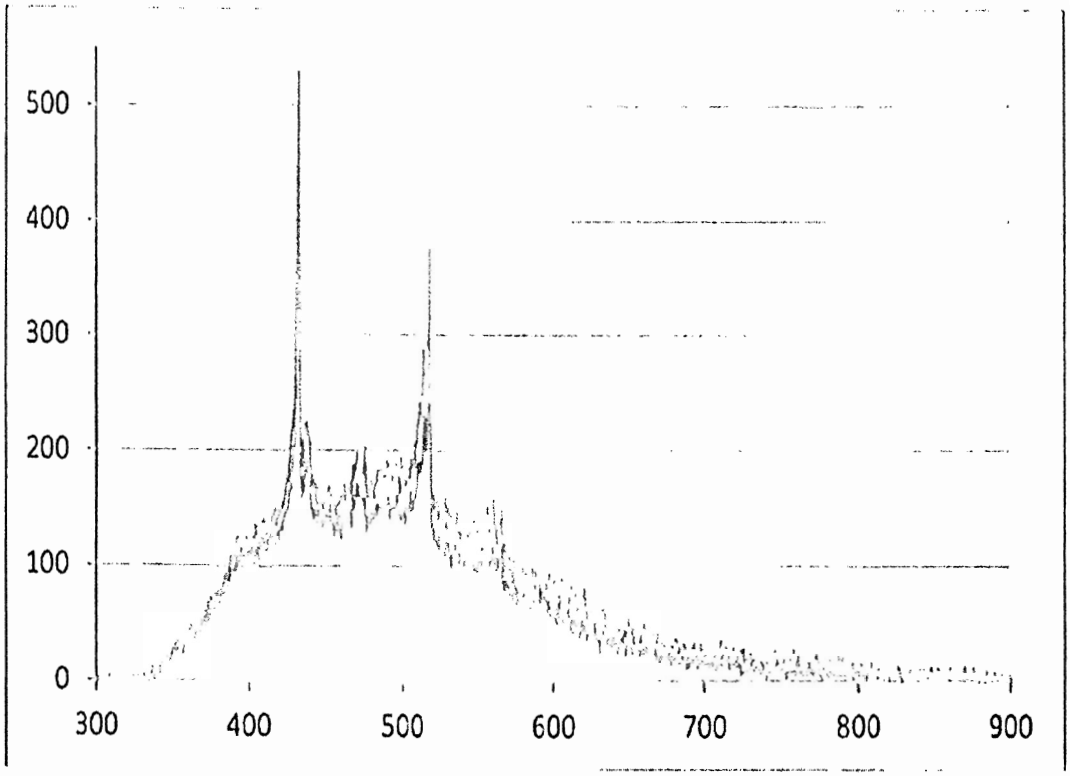


Figure 14: Magnegas from crude oil shows higher intensity of flame only and higher emission lines (red colour) when we go from center of the flame to 2 mm from the center

Two kinds of flames had been subjected to experimentation. Magnegas flame from antifreeze was much higher than that of crude oil. Magnegas has been known to have very fast metal cutting capabilities. It is been tested against other competing gases currently used in the market to cut metal, like using Acytelene, magnegas promises to be a gas with various applications in energy, powering automobiles and other applications.Figure 9(a) and 9(b) represent the experimental setup of experiment. A 10 mm focusing lens was mounted perpendicular to the flame from magnegas at 5 mm from the flame source. The lens was connected to a high resolution fibre optic spectrometer ocean optics HR2000 through a fibre optic cable. The spectrometer provides a optical resolution to 0.035 nm and going up. The nozzle is alignment was done with a laser pen aimed at the centre axis of the flame in the z direction. Measurement was done on the flame by taking a circular slice and going recording of data was taken at time t=0 when cutting of metal began and continuously afterward data was collected at intervals of 50 ms until cutting was complete at approximately 3-5 second depending of the size of the metal cut. From the centre to the outer perimeter of the flame in the horizontal y direction. magnegas from crude oil was kept at 15 Psi pressure, the one from antifreeze was at 20 Psi pressure and oxygen at 40 Psi pressure. First experiment consisted of combining magnegas from the crude oil and oxygen together. The second was combining magnegas from antifreeze to oxygen. The mixing of magngas and oxygen occurred at the tip of the Harris nozzle. For the measurement of magngas flame while cutting, a scanning method was used. Each scan of 50 ms produced a graph of the electromagnetic spectrum from the flame at that particular time later all the graphs where compared for temperature and emission lines. Emission lines were observed in the flames from both magnegas from crude oil and antifreeze. The peak wavelength of magnegas from crude oil flame only is λmax =473 nm and the peak wavelength while cutting metal is λmax =739 nm, that is a 266 nm difference. The difference for magnegas from antifreeze was 259 nm; λmax =476 nm for flame only and λmax =735 nm for flame while cutting metal. The reason for the difference is due to interaction of the flame with metal when the flame is positioned very close to the metal (2-4 mm) in order to cut it. This resulted in a spectrum of flame/metal in the red range. When no metal was present, the flame displayed a blue colour. Figure 10

and 12 represents the Magnegas from crude oil and antifreeze, flame only,3 mm from the base. Figure 11 and 13 depict the flame of magnegas from crude oil and antifreeze at 3 mm above the surface of metal while cutting it. The Intensity peaked as we went up away from the base of the flame for the first 2-4 mm, started decreasing as we kept getting away from the base. Maximum effect of the flame lays directly above the base and gets weaker as we move away from it. Intensity from magnegas antifreeze was 15-20 times higher than that of crude oil shown in figure 14. The reason is unknown, but one factor for higher intensity could be that magnegas antifreeze was kept at 20 Psi while magnegas crude oil was at 15 Psi. It is 33.3 % increase in pressure . The 33.3 % pressure increase could not alone justify the over 1500 % increase in intensity. Results are tabulated in table 2 and also shown in figures 10-14.

Table 2: Summary of the measured emission lines.

Emission lines (in nano meters)	**magngas crude** oil flame only	**magngas crude** oil cutting metal	**magnegas antifreeze** flame only	**magnegas antifreeze** cutting metal
CH and CN	391	391	391.2	
CH	433	432	430.77	432.15
CH	468	468		
C2	518	518	430.77, 469.71, 518.48, 556.11, 600	517.68
Unidentified	577	590		590.42
Unidentified	594	768		767.68

Unidentified peak observed at 590 nm and 768 nm which confirmed presence of new species known as magnegas. The temperature of the both the flames was computed using Weins Law for the peak of each flames spectrum. The magnegas flames were evaluated at 10,500 F in temperature. The distribution is non-black body like. non H emission lines were visible which indicates unknown elements in magnegas.

6 Applications of Magnegas

6.1 Use of magnegas for the generation of electricity

Fossil fueled electric power plants release the largest source of pollution in our atmosphere today. Most of the alarming environmental problems are caused precisely by the combustion of fossil fuels in electric power plants. Use of magnegas as additive to fossil fuels in electric power plants as excellent application of magnegas is that as an additive to any fossil fuel for the production of electricity in current power plants. When Magnegas added in plant furnace as addative, then Magnegas mixes very well with natural gas. Magnegas can be injected anywhere in natural gas pipelines or injected directly in the plant furnace. When electric power plants burn oil or coal, reduced the pollutant and resulting advantages, such as: (1) Improved quality of the exhaust (2) Improved efficiency (3) Pre-setting the exhaust quality as required by the EPA, and then computing the percentage of magnegas needed for its achievement. Magnegas can reduce the alarming oxygen depletion caused by fossil fueled power plants, estimated in the permanent removal of about five millions metric tons of breathable oxygen per day.

6.2 Use of Magnegas and magnehydrogen as automotive fuel

The achievement of a 100% American clean fuel, is today a reality because Magnegas can now be produced in any desired quantity, in any desired location, at any desired time, while being cleaner, safer and cheaper then gasoline and other fossil fuels.This is possible only because of Professor Santilli. It is important to note that car capable of operating on Magnegas are already in production and use today all over the world, and are given by cars produced to operate on Compressed Natural Gas (CNG). In fact, a brand new Honda Civic built to operate on natural gas was purchased; natural gas was removed from the car; the tank was filled-up with magnegas and the car operated, thus proving that magnegas and natural gas are interchangeable. Equally excellent results were obtained by using a 50-50 mixture of Magnegas and natural gas (also operate without catalytic con-

verter). The conversion of gasoline fueled cars to operate on Magnegas is simple and consists in: adding a pressure tank in the trunk, adding a fuel line; and installing a pressure regulator releasing magnegas into the intake manifold. In this case the car becomes dual, namely it can operate either with gasoline or Magnegas. The transition from one fuel to the other is possible with a switch even while driving (brown Honda Civic car in figure 15). The car can also be operated via the joint combustion of Magnegas and gasoline with a dramatic improvement of the exhaust. The safety of magnegas over gasoline should also be noted. Gasoline is, by far, the most dangerous fuel used by mankind, because gasoline tanks are paper thin, and often rupture in car accidents, in which case gasoline spills and explodes. By comparison, high pressure magnegas tanks are dramatically safer (they are tested with guns) and cannot rupture under any possible car accident. In the event a magnegas fuel line (rather than the tank) is broken by an accident, magnegas is lighter than air, rapidly disperses in the atmosphere, and does not spill. In the event of a flame near a ruptured line, magnegas burns without explosion (because combustion cannot propagate inside the tank due to the lack of oxygen, the fuel being gaseous -not so for a liquid fuel such as gasoline due to the air over the liquid line). Also, gasoline refills are slow and dangerous, particularly for race cars, while the refill of magnegas is virtually instantaneous and safer, again, because magnegas is lighter than air. Finally, being a gas, magnegas weighs less than gasoline, thus permitting a better performance. Figure 15 represent a view of the two Honda Civic used for extensive tests on magnegas.

6.3 MagneGas as a cooking or heating gas - safe for indoor use

Carbon Monoxide (CO) is one of the most dangerous poisons gas produced by incomplete combustion of fuels i.e gasoline, kerosene, natural gas, propane and liquefied petroleum gas. It replaces oxygen in the blood stream causing suffocation. The results of breathing CO include loss of function, brain damage and even death. The greatest danger occurs when fossil fuels are burned indoors where they can concentrate and amplify the damage that they cause. Santilli MagneGas is safe for indoor use. MagneGas offers five

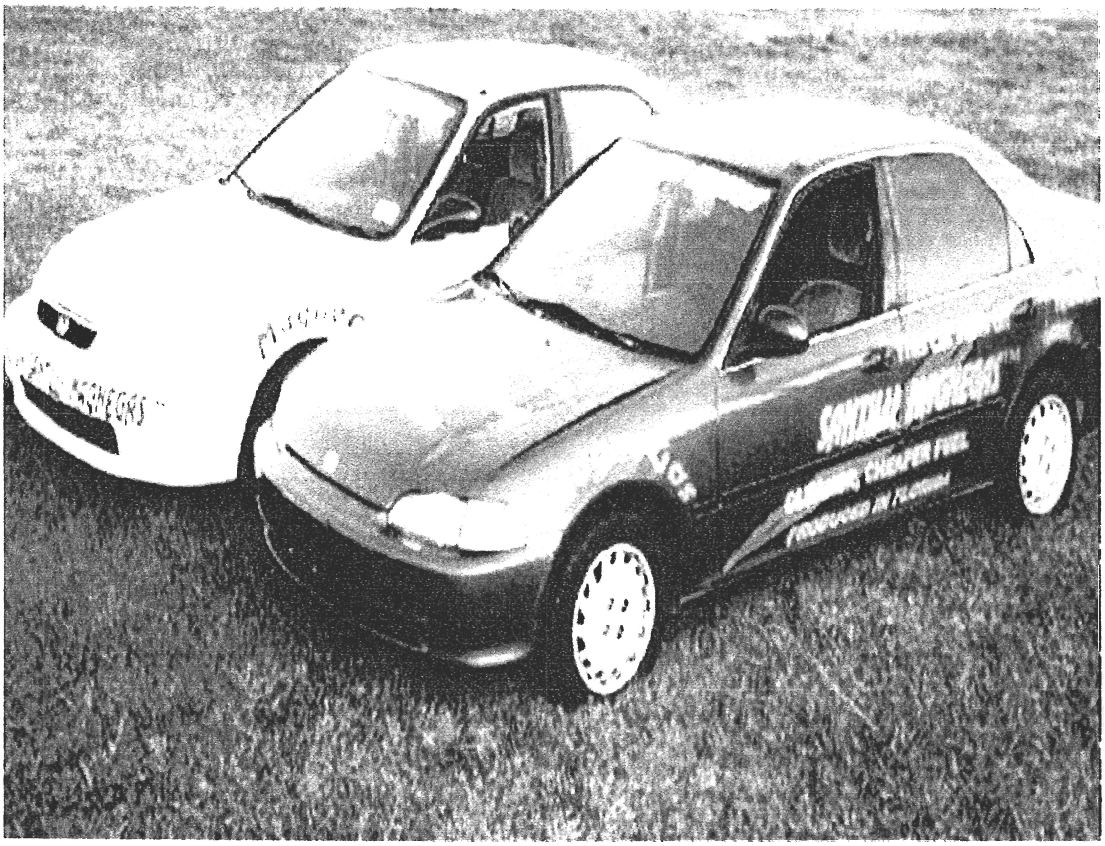


Figure 15: A view of the two Honda Civic used for extensive tests on magnegas.

major safety advantages over fossil fuels such as natural gas and propane:

1. Magnegas exhaust is comprised of water vapor, oxygen and carbon dioxide. Because its exhaust is free of toxic fumes and carcinogens, it is safe for indoor use.

2. Because Magnegas exhaust has a positive oxygen balance, its does not present the dangers of oxygen depletion in closed environments that other fuels do.

3. Magnegas is lighter than air, substantially reducing the danger of explosion compared to fossil fuels, that are heavier than air and tend to pool in and around work areas. Although it contains as much or more energy than fossil fuels, it is also less explosive.

4. Although Magnegas is clear, it has a natural odour allowing it to be easily detected.

5. Magnegas is magnetic, it can be easily compressed offering the same portability as propane.

6.4 Use of Magnegas for metal cutting

Acetylene has been traditionally used as a metal cutting gas. The extremely hot gases created by burning acetylene in combination with oxygen allow thick metals such as steel to be cut quickly. Acetylene is highly regulated by the U.S. EPA (Environmental Protection Agency) and OSHA (Occupational Safety and Health Administration). This regulatory environment restricts its production, transport and use by making permits for new production facilities very difficult to get. These additional restrictions have been implemented because acetylene is dangerous to make and transport since it is unstable and potentially explosive.

The interim replacement for acetylene is liquefied petroleum gas, or LPG. LPG is often combined with Methylacetylene-Propadiene to create MAPP gas. LPG and MAPP are often equally as expensive as acetylene because production facilities are few. MagneGas offers five major advantages as an alternative to acetylene, MAPP and LPG as a cutting gas. MagneGas offers significant safety features:

1. Comprised of water vapor, oxygen and carbon dioxide, its exhaust is free of toxic fumes and carcinogens making it safe for indoor use. Because MagneGas exhaust has a positive oxygen balance, its does not present the dangers of oxygen depletion in closed environments that other fuels do.

2. Compared to fossil fuels, MagneGas is less explosive making it much safer to use.

3. MagneGas is lighter than air, substantially reducing the danger of explosion compared to fossil fuels, which are heavier than air and tend to pool in and around work areas.

4. Although MagneGas is clear, it has a natural odor allowing it to be easily detected.

5. Magnegas Generators are equipped with safety devices and automatic shut-offs to ensure that the gas generation process is as safe as the gas itself.

Compared to acetylene, MagneGas is very cost competitive even after considering cost of equipment, consumables, electricity and labor.

7 Conclusions

In preceding sections, we have described adequate details of the important application of hadronic chemistry in the production of magnegas. The aspects those covered herein are the basics of magnecular interactions and some applications of magnegases. The magnecules are pollution free fuels because it provides stored magnetic energy without the molecular bond cleavage. Magnegases find application in metal cutting process. MagneGas offers major advantages as an alternative to acetylene, MAPP and LPG as a cutting gas. Magnegases are marvellous energy suppliers for powering automobiles. The Flame temperature report of the new magnecular species of gases that actually emphasizes the importance of this new field [24-25].

8 Acknowledgments

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METHODS OF NONCOMMUTATIVE GEOMETRY WITH APPLICATIONS TO ASTROPHYSICS AND COSMOLOGY*

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Abstract

In these lecture notes, we want to present a selfcontained introduction to certain aspects of noncommutative geometry and some applications in astrophysics. The concepts of noncommutative spaces and Hopf algebra symmetries are discussed, with special emphasis on twist deformation. Based on that, twisted gravity is introduced and some of its applications are discussed. So we provide explicit examples for noncommutative Einstein spaces and discuss an explicit black hole solution.

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1 Introduction

In noncommutative geometry, coordinates are promoted to operators with non-vanishing commutation relations:

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\Theta^{ij}(\hat{x})\,, \tag{1}$$

where $\Theta^{ij} = -\Theta^{ji}$ satisfiy the Jacobi identity:

$$[\Theta^{ij}(\hat{x}),\, \hat{x}^k] + [\Theta^{ki}(\hat{x}),\, \hat{x}^j] + [\Theta^{jk}(\hat{x}),\, \hat{x}^i] = 0\,. \tag{2}$$

The aim is to build models on such space-time structures and study their properties. In this talk, we want to provide various motivations for studying noncommutative geometries and to introduce the basic mathematical notions. What is most remarkable in my eyes, is the intimate connection between noncommutative gauge theory and gravity. This connection is not fully understood at present and studied from different points of view, see e.g. [1–6] and references therein for a merely exemplary list of citations. In this talk, we want to review some approaches to noncommutative gravity and applications to astrophysics.

Let us start with some historical remarks, since noncommutative spaces do have quite a history. Even in the early days of quantum mechanics and quantum field theory, continuous space-time and Lorentz symmetry were considered inappropriate to describe the small scale structure of the universe [7]. It was also argued that one should introduce a fundamental length scale limiting the precision of position measurements. In [8, 9] the introduction of a fundamental length is suggested to cure the ultraviolet divergencies occuring in quantum field theory. H. Snyder was the first to formulate these ideas mathematically [10]. He introduced commutation relations closing in the Lorentz algebra. A position uncertainty is inherent. The success of the renormalisation programme made people forget about these ideas for some time. But when the quantisation of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate and that space-time has to be quantised or noncommutative, in some way.

There is a deep conceptual difference between quantum field theory and gravity: In the former space and time are considered as parameters, in the

latter as dynamical entities. In order to combine quantum theory and gravitation (geometry), one has to describe both in the same language, this is the language of algebras [11]. Geometry can be formulated algebraically in terms of abelian $\mathbf{C}^*$ algebras and can be generalised to non-abelian $\mathbf{C}^*$ algebras (noncommutative geometry). A further abstraction leads to the formulation in terms of double bicategories, see e.g. [12] and references therein.

Quantised gravity may even act as a regulator of quantum field theories. This is encouraged by the fact that noncommutative geometry introduces a lower limit for the precision of position measurements. There is a very nice argument showing that, on a classical level, the selfenergy of a point particle is regularised by the presence of gravity [13]:

Gravitational self energy

Let us consider an electron and a fictitious shell of radius ϵ around the electron. We identify the selfenergy of the electron with the selfenergy of the shell $m(\epsilon)$, in the limit $\epsilon \to 0$. $m(\epsilon)$ is given by

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon},$$

where m_0 is the restmass and e the charge of the electron. In the limit $\epsilon \to 0$, $m(\epsilon)$ will diverge. Including Newtonian gravity we have to modify this equation:

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm_0^2}{\epsilon},$$

G denotes Newton's gravitational constant. Thus, $m(\epsilon)$ will still diverge for $\epsilon \to 0$, unless the mass and the charge are fine tuned. Considering general relativity, we know that energy, therefore also the energy of the electron's electric field, is the source of a gravitational field. Again, we have to modify the above equation:

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm(\epsilon)^2}{\epsilon}.$$

The solution of this quadratic equation is straight forward:

$$m(\epsilon) = -\frac{\epsilon}{2G} \pm \frac{\epsilon}{2G}\sqrt{1 + \frac{4G}{\epsilon}(m_0 + \frac{e^2}{\epsilon})}.$$

We are interested in the positive root. Miraculously, the limit $\epsilon \to 0$ is finite,

$$m(\epsilon \to 0) = \frac{e}{G}.$$

This is a non-perturbative result, since $m(\epsilon \to 0)$ cannot be expanded around $G = 0$. $m(\epsilon \to 0)$ does not depend on m_0, therefore there is no fine tuning present. Classical gravity regularises the selfenergy of the electron. However, this does not make the quantisation of space-time unnecessary, since quantum corrections will destroy the above picture. But it provides an example for the regularisation of physical quantities by introducing gravity. So hope is raised that the introduction of gravity formulated in terms of noncommutative geometry may regularise physical quantities even on the quantum level.

Landau problem

Here, noncommutative coordinates arise classically in a certain limit. Let us consider a particle with charge e moving in a plane with a magnetic field normal to that plane. This situation is described by the following action:

$$S = \int dt(\frac{1}{2}m\dot{x}_i\dot{x}^i - \frac{e}{c}B_{ij}x^i\dot{x}^j)\,, \tag{3}$$

where $B_{ij} = -B_{ji} = \text{const}$. In the limit of strong magnetic field, we project onto the lowest Landau level. As a result, we obtain a non-vanishing Dirac-bracket

$$\{x^i, x^j\}_{\text{DB}} = \frac{c(B^{-1})^{ij}}{e}\,. \tag{4}$$

Quantum gravity

Measuring a distance l between two particles, we have to use photons of appropriate wavelength. An energy $E = \frac{hc}{\lambda} \sim \frac{hc}{l}$ has to be deposited in the measuring region. If l is of the order of the Planck length l_{Pl},

$$l \sim l_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}}\quad, \tag{5}$$

the energy deposited is given by

$$E = hc\sqrt{\frac{c^3}{\hbar G}}\,. \tag{6}$$

This corresponds to an inertia of the Planck mass:

$$M = \frac{E}{c^2} = \frac{h}{c}\sqrt{\frac{c^3}{\hbar G}} \sim \sqrt{\frac{\hbar c}{G}} = M_{Pl}\,.$$

The associated Schwarzschild radius is given by $r_S = \frac{2GM}{c^2} = 2\sqrt{\frac{\hbar G}{c^3}} = 2l_{Pl}$. This means that the measuring process creates a black hole. In order to avoid this scenario, uncertainty relations for coordinates have to be assumed [14]. A natural way to achieve uncertainty relations is via Eq. (1).

String theory

In the context of open strings with D-branes in a background magnetic field $B_{\mu\nu}$, the end points of the open strings are confined to the D-branes. Space-time dynamics is given by a noncommutative field theory, in the low energy limit, see e.g. [15–17].

2 Coordinate algebra

Let us discuss the commutation relations for the coordinates,

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\Theta^{ij}(\hat{x})\,,$$

in more detail. The structure constants $\Theta^{ij}(\hat{x})$ might be any function of the generators with $\Theta^{ij} = -\Theta^{ji}$ and satisfying the Jacobi identity. Most commonly, the commutation relations are chosen to be either constant, linear or quadratic in the generators. In the canonical case the relations are constant,

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{\mathrm{ij}} = \mathrm{const}\,. \tag{7}$$

The linear or Lie-algebra case,

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\lambda^{\mathrm{ij}}_{\mathrm{k}}\hat{\mathrm{x}}^{\mathrm{k}}\,, \tag{8}$$

where $\lambda_k^{ij} \in \mathbb{C}$ are the structure constants, basically has been discussed in two different settings, namely fuzzy spaces and κ-deformation. The third commonly used choice is a quadratic commutation relation,

$$[\hat{x}^i, \hat{x}^j] = (\frac{1}{q}\hat{R}^{ij}_{kl} - \delta^i_l \delta^j_k)\, \hat{x}^k \hat{x}^l \,, \tag{9}$$

where $\hat{R}^{ij}_{kl} \in \mathbb{C}$ is the so-called $\hat{R}$-matrix corresponding to quantum groups and is a solution of the Quantum-Yang-Baxter-Equation.

Independent of the explicit form of Θ^{ij}, the commutative algebra of functions on ordinary space-time has to be replaced by the noncommutative algebra $\widehat{\mathcal{A}}$ generated by the coordinates $\hat{x}^i$, subject to the ideal $\mathcal{I}$ of relations generated by the commutation relations,

$$\widehat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ...\hat{x}^n \rangle\rangle}{[\hat{x}^i, \hat{x}^j] - \mathrm{i}\Theta^{ij}} \,. \tag{10}$$

A noncommutative function $\hat{f}$ is given by a (formal) power series

$$\widehat{\mathcal{A}} \ni \hat{f}(\hat{x}) = \sum_{n=0}^{\infty} c_{i_1 \dots i_n} : \hat{x}^{i_1} \dots \hat{x}^{i_n} : , \tag{11}$$

where :: indicates an ordering recipe for monomials in $\widehat{\mathcal{A}}$. *Normal ordering* e.g. means that coordinates with lower indices are moved to the left, i.e.

$$: \hat{x}^1 \hat{x}^2 \hat{x}^1 := (\hat{x}^1)^2 \hat{x}^2 \,, \tag{12}$$

or in case of *symmetrical ordering*:

$$: \hat{x}^1 \hat{x}^2 \hat{x}^1 := \frac{1}{3}((\hat{x}^1)^2 \hat{x}^2 + \hat{x}^1 \hat{x}^2 \hat{x}^1 + \hat{x}^2 (\hat{x}^1)^2) \,. \tag{13}$$

In the latter situation, we can also use plane waves,

$$\mathrm{e}^{\mathrm{i}k_j \hat{x}^j} \,,$$

as basis. Let $\mathcal{A}$ be the commutative algebra of functions. Then, there is an isomorphism mapping from the commutative function algebra equipped

with an additional noncommutative product $\star$ to the noncommutative one $\widehat{\mathcal{A}}$,

$$W : (\mathcal{A}, \star) \longrightarrow (\widehat{\mathcal{A}}, \cdot) .$$

This isomorphism exists, iff the noncommutative algebra together with the chosen basis (ordering) satisfies the so-called Poincaré-Birkhoff-Witt property, i.e. any monomial of order n can be written as a sum of the basis monomials of order n or smaller, by reordering according to the algebra relations and using the Campbell-Baker-Hausdorff formula.

Let us stick to the most important example: the canonical deformation with $\Theta^{ij}(\hat{x}) = \theta^{ij} = const$ and symmetric ordering: The so-called Weyl-map is given by

$$\begin{aligned} W : f(x) \mapsto \hat{f}(\hat{x}) &:= \int d^4k \tilde{f}(k) \mathrm{e}^{\mathrm{i}k_j \hat{x}^j} , \\ \text{where} \qquad \tilde{f}(k) &= \tfrac{1}{(2\pi)^4} \int d^4x f(x) \mathrm{e}^{-\mathrm{i}k_j x^j} \end{aligned} \tag{14}$$

is the usual Fourier transform of $f(x)$. Plane waves are mapped onto plane waves,

$$W : \mathrm{e}^{\mathrm{i}k_j x^j} \mapsto \mathrm{e}^{\mathrm{i}k_j \hat{x}^j} .$$

First of all this means that the map W establishes an isomorphism of vector spaces. In order to extend W to an algebra isomorphism, we have to introduce a new noncommutative multiplication $\star$ in $\mathcal{A}$. This star product is defined by

$$W(f \star g) := W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{15}$$

where $f, g \in \mathcal{A}$, $\hat{f}, \hat{g} \in \widehat{\mathcal{A}}$. We obtain

$$\begin{aligned} W(f)W(g) &= \int d^4k d^4p \tilde{f}(k) \tilde{g}(p) \mathrm{e}^{\mathrm{i}k_j \hat{x}^j} \mathrm{e}^{\mathrm{i}p_j \hat{x}^j} \\ &= \int d^4k d^4p \tilde{f}(k) \tilde{g}(p) \mathrm{e}^{-\frac{\mathrm{i}}{2} p_m \theta_{mn} k_n} \mathrm{e}^{\mathrm{i}(k_j + p_j)\hat{x}^j} , \end{aligned} \tag{16}$$

and applying W^{-1} yields

$$\begin{aligned} f \star g\,(x) &= \int d^4k d^4p \tilde{f}(k) \tilde{g}(p) \mathrm{e}^{-\frac{\mathrm{i}}{2} p_m \theta_{mn} k_n} \mathrm{e}^{\mathrm{i}(k_j + p_j)x^j} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{ij}\partial_i^x \partial_j^y} f(x)\, g(y)\Big|_{y \to x} . \end{aligned} \tag{17}$$

This is the so-called Moyal-Weyl star product, and W is an algebra isomorphism by construction,

$$(\mathcal{A}, \star) \cong (\widehat{\mathcal{A}}, \cdot)\,. \tag{18}$$

The information about the noncommutativity of $\widehat{\mathcal{A}}$ is encoded in the star product.

2.1 Twists

The Weyl-Moyal case is also the simplest example for twist deformation:

$$f \star g(x) = m \circ \mathcal{F}^{-1}(f \otimes g)\,, \tag{19}$$

where $\mathcal{F} = e^{-\frac{1}{2}\theta^{ij}\partial_i \otimes \partial_j}$ denotes the so-called Drinfel'd twist for canonical deformation. Let us introduce the useful Sweedler-notation:

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \qquad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha\,, \tag{20}$$

where f_α, f^α are some vector fields, and we imply a sum over α. With this notation, we have

$$f \star g(x) = \bar{f}^\alpha(f)\, \bar{f}_\alpha(g)\,. \tag{21}$$

The so-called $\mathcal{R}$ matrix is defined by

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1} = R^\alpha \otimes R_\alpha,\ \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha\,, \tag{22}$$

with $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$. The $\mathcal{R}$ matrix "measures" the noncommutativity:

$$f \star g = \bar{R}^\alpha(g) \star \bar{R}_\alpha(f)\,. \tag{23}$$

Even more: The twist deforms the classical symmetry group $\mathcal{G}$ of the space – considered as the Hopf algebra $\mathcal{U}g = (h, \Delta, \epsilon, S)$, which is the universal enveloping algebra of its Lie algebra g. A Hopf algebra possesses dual structures. On the one hand, it is an algebra; on the other hand it has a co-algebra structure consisting of a co-product Δ (dual to the normal multiplication), a so-called antipode and a co-unit. These structure maps need to satisfy consistency conditions.

But, we are proceeding too fast. Let us step back a bit and formulate the definition of a Hopf algebra with some care: A Hopf algebra A (over $\mathbb{C}$)

consists of an algebra and a co-algebra structure which are compatible with each other. Additionally, there is a map called antipode, which corresponds to the inverse of a group. A is an algebra, i.e., there is a multiplication m and a unit element η,

$$\begin{aligned} m &: A \otimes A \to A, \\ & \quad a \otimes b \mapsto ab, \\ \eta &: \mathbb{C} \to A, \\ & \quad c \mapsto c\,\mathbf{1}_A, \end{aligned}$$

such that the multiplication satisfies the associativity axiom (Fig. 1), where id denotes the identity map, and η the axiom depicted in Fig. 2. Reversing

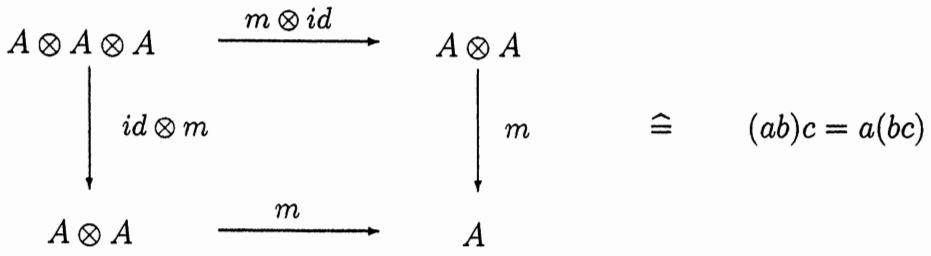

Figure 1: Associativity

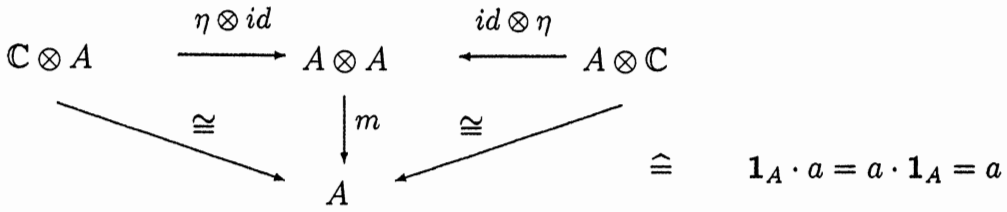

Figure 2: Unity axiom

all the arrows in Figs. 1 and 2 and replacing m by the so-called co-product Δ and η by the co-unit ϵ gives us the axioms for the structure maps of the

co-algebra. The co-product and the co-unit,

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A, \\ \epsilon : A &\rightarrow \mathbb{C} \end{aligned}$$

are dual to m and η, respectively. Compatibility between algebra and co-algebra structure means that the co-product Δ and the co-unit ϵ are algebra homomorphisms, i.e.,

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{24}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \tag{25}$$

where $a, b \in A$. The antipode $S : A \rightarrow A$ satisfies the axiom shown in Fig. 3 below. It is an anti-algebra homomorphism ($S(ab) = S(b)S(a)$). Associativity on the algebra side is related to *co-associativity* on the dual

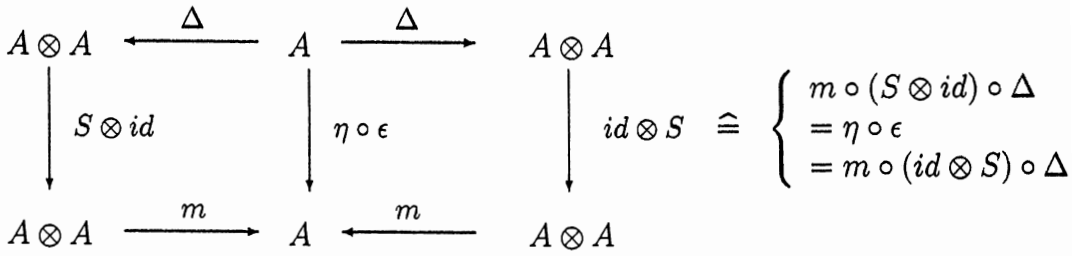

Figure 3: Antipode axiom

co-algebra side, i.e.

$$(ab)c = a(bc) \qquad \Rightarrow id \otimes \Delta\,(\Delta(a)) = \Delta \otimes id\,(\Delta(a))\,. \tag{26}$$

For our purposes, the co-product Δ is most important. In case of the universal enveloping algebra of a Lie algebra g, the co-product is given by

$$\Delta h = h \otimes \mathbf{1} + \mathbf{1} \otimes h\,, \quad h \in g\,. \tag{27}$$

In the case of twisted symmetries, this co-product is modified in the following way:

$$\begin{aligned} \Delta : \mathcal{U}g &\rightarrow \mathcal{U}g \otimes \mathcal{U}g\,, \\ \Delta &\rightarrow \Delta_{\mathcal{F}} = \mathcal{F}\,\Delta\,\mathcal{F}^{-1}\,, \\ \mathcal{F} &\in \mathcal{U}g \otimes \mathcal{U}g\,, \end{aligned} \tag{28}$$

using the twist defined in (20). It has to satisfy the following conditions (which guarantee associativity of the $\star$-product):

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}\,, \tag{29}$$

$$(\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}\,. \tag{30}$$

Using the twist, all kind of products can be deformed [18, 19]. The star product between functions we had already before:

$$f \star g = \bar{f}^{\alpha}(f)\, \bar{f}_{\alpha}(g)\,.$$

In the same way, we can consistently deform the product between a function h and a vector field v

$$h \star v \equiv \bar{f}^{\alpha}(h)\, \bar{f}_{\alpha}(v)\,, \tag{31}$$

or between tensor fields τ, τ':

$$\tau \otimes_{\star} \tau' := \bar{f}^{\alpha}(\tau) \otimes \bar{f}_{\alpha}(\tau')\,, \tag{32}$$

where $\bar{f}^{\alpha}(\cdot)$ is understood as the classical Lie derivative with respect to the vector field $\bar{f}^{\alpha}$. It is now the strategy is to deform any product $\circ$ of elements A and B of some representation of $\mathcal{H}$ by replacing it with the twisted product $\circ_{\star}$:

$$A \circ B \mapsto A \circ_{\star} B := \bar{f}^{\alpha}(A) \circ \bar{f}_{\alpha}(B)\,. \tag{33}$$

3 Twisted gravity

Using the technique of twist deformation introduced before, let us define the $\star$-Lie derivative, i.e., infinitesimal diffeomorphisms:

$$\mathcal{L}^{\star}_{u}(h) := \bar{f}^{\alpha}(u)(\bar{f}_{\alpha}(h))\,. \tag{34}$$

The space of vectorfields equipped with the $\star$-Lie bracket

$$\mathcal{L}^{\star}_{u}\mathcal{L}^{\star}_{v} - \mathcal{L}^{\star}_{\bar{R}^{\alpha}(v)}\mathcal{L}^{\star}_{\bar{R}_{\alpha}(u)} =: \mathcal{L}^{\star}_{[u,v]_{\star}}\,, \tag{35}$$

where $[u,v]_\star = u\star v - \hat{R}^\alpha(v)\star\hat{R}_\alpha(u)$, defines the $\star$-Lie algebra of vector fields, denoted by $\Xi_\star$. Following [18, 19], we can introduce the notions of twisted Riemannian geometry. The star-connection $\nabla^\star$

$$\nabla^\star_u : \Xi_\star \to \Xi_\star\,,$$

is defined to satisfy the following axioms:

$$\begin{aligned}\nabla^\star_{u+v}z &= \nabla^\star_u z + \nabla^\star_v z\,,\\ \nabla^\star_{h\star u}v &= h\star\nabla^\star_u v\,,\\ \nabla^\star_u(h\star v) &= \mathcal{L}^\star_u(h)\star v + \bar{R}^\alpha(h)\star\nabla^\star_{\bar{R}_\alpha(u)}v\,,\end{aligned} \tag{36}$$

where $u,v,z\in\Xi_\star$, h is a function. Next, we define connection coefficients by

$$\nabla^\star_\mu\hat{\partial}_\nu := \Gamma^\sigma_{\mu\nu}\star\hat{\partial}_\sigma\,,$$

using the basis $\{\hat{\partial}_\mu\}$ of $\Xi_\star$. The action of the covariant derivative on a one-form can be obtained by employing the star-dual pairing of a vector field v with a one-form ω,

$$\nabla^\star_u < v,w >_\star = \mathcal{L}^\star_u < v,w >_\star = < \nabla^\star_u v, w >_\star + < \bar{R}^\alpha(v), \nabla^\star_{\bar{R}_\alpha(u)}w >_\star, \tag{37}$$

which equivalently can be written as

$$< v, \nabla^\star_u w >_\star = \mathcal{L}_{\bar{R}^\alpha(u)} < \bar{R}_\alpha(v), w >_\star - < \nabla^\star_{\bar{R}^\alpha(u)}(\bar{R}_\alpha(v)), w >_\star\ . \tag{38}$$

The definition of the star-dual pairing is along the lines of (33):

$$< v,w >_\star := < \bar{f}^\alpha(v), \bar{f}_\alpha(w) >\ . \tag{39}$$

The torsion and curvature tensors are given by

$$T^\star(u,v) = \nabla^\star_u v - \nabla^\star_{\bar{R}^\alpha(v)}\bar{R}_\alpha(u) - [u,v]_\star, \tag{40}$$

$$\begin{aligned}R^\star(u,v,z) &\equiv R^\star(u,v)z\\ &= \nabla^\star_u\nabla^\star_v z - \nabla^\star_{\bar{R}^\alpha(v)}\nabla^\star_{\bar{R}_\alpha(u)}z - \nabla^\star_{[u,v]_\star}z\,.\end{aligned} \tag{41}$$

It is enough to calculate the components of the tensor with respect to the basis $\{\hat{\partial}_i\}$ because of the tensorial property, i.e.,

$$T^\star(u,v) = u^i\star T^\star(\hat{\partial}_i,\hat{\partial}_j)\star v^j. \tag{42}$$

In this frame, the star-connection is given by

$$\nabla^\star_z u = \mathcal{L}^\star_z(u^i) \star \hat{\partial}_i + \bar{R}^\alpha(u^i) \star \bar{R}_\alpha(z)^j \star \Gamma^k_{ji} \star \hat{\partial}_k \,. \tag{43}$$

In components, the torsion and curvature are given by:

$$\begin{aligned} T(\hat{\partial}_\mu, \hat{\partial}_\nu) &= T_{\mu\nu}{}^\rho \hat{\partial}_\rho \,, \\ R(\hat{\partial}_\mu, \hat{\partial}_\nu, \hat{\partial}_\rho) &= R_{\mu\nu\rho}{}^\sigma \hat{\partial}_\sigma \,. \end{aligned} \tag{44}$$

As in the commutative case, the Ricci tensor is defined by the following contraction of indices:

$$Ric_{\mu\nu} = R_{\rho\mu\nu}{}^\rho \tag{45}$$

The metric is a $\star$-symmetric tensor

$$g^\star = g^\star_{\mu\nu} \star d\hat{x}^\mu \otimes_\star d\hat{x}^\nu \,.$$

The components of the inverse metric are determined with respect to the star product,

$$g^{\star\mu\nu} \star g^\star_{\nu\rho} = g^\star_{\rho\nu} \star g^{\star\nu\mu} = \delta^\mu_\nu \,.$$

A noncommutative metric is said to be compatible with the star-connection, if

$$\nabla^\star_u g^\star = 0 \,, \forall u \in \Xi_\star \,.$$

Therefore, given a metric $g^\star$ one can consistently define Christoffel symbols $\Gamma^{\star i}_{jk}$, a compatible connection $\nabla^\star$, Riemann $R^\star_{ijk}{}^l$ and Ricci tensors $Ric^\star_{ab}$.

Then, the deformed Einstein equation has the familiar form

$$R^\star_{ab} - \frac{1}{2} g^\star_{ab} \star R^\star = 8\pi G_N T^\star_{ab} \,. \tag{46}$$

Of special interest are Einstein spaces, where the deformed Ricci tensor is proportional to the noncommutative metric,

$$R^\star_{\mu\nu} = c\, g^\star_{\mu\nu} \,. \tag{47}$$

4 Noncommutative Einstein spaces

In this section, we will review the results of [20]. The concepts of twisted geometry, which we have introduced in the last section, will be applied to various examples. We will first discuss the Weyl-Moyal plane and the q-deformed 5-dimensional plane, then twisted spheres, based on the q-deformed plane. The last example is the 2-dimensional Weyl-Moyal plane with a non-trivial noncommutative metric.

4.1 Weyl-Moyal plane $\mathbb{R}^4_\theta$

The Weyl-Moyal plane is defined by the following Abelian twist:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j}\,, \tag{48}$$

where $\theta^{ij} = -\theta^{ji} \in \mathbb{R}$. Remembering the definitions of the last section, we obtain for the covariant derivative

$$\nabla^\star_z u = z^i \star \partial_(u^j) \star \partial_j + z^i \star u^j \star \Gamma^{\star k}_{ij} \star \partial_k\,. \tag{49}$$

In a first step, let us show that the choice $\Gamma^{\star k}_{ij} = 0$ is consistent with the axioms (36) and furthermore defines a Levi-Civita connection. With our choice for the deformed connection coefficients, the expression for the covariant derivative (49) becomes

$$\nabla^\star_z u = z^i \star \partial_i(u^j) \star \partial_j\,. \tag{50}$$

Let us show that the axioms (36) are satisfied:

- $$\nabla^\star_{u+v} z = (u+v)^i \star \partial_i(z^j) \star \partial_j = \nabla^\star_u z + \nabla^\star_v z\,. \tag{51}$$

- $$\begin{aligned} \nabla^\star_{h\star u} v &= (h \star u^i) \star \partial_i(v^j) \star \partial_j \\ &= h \star (u^i \star \partial_i v^j \star \partial_j) = h \star \nabla^\star_u v\,. \end{aligned} \tag{52}$$

$$\begin{aligned}\nabla^\star_u(h\star v) &= u^i\star\partial_i(h\star v^j)\star\partial_j \\ &= \mathcal{L}^\star_u(h)\star v + u^i\star h\star(\partial_i v^j)\star\partial_j \\ &= \mathcal{L}^\star_u(h)\star v + \bar{R}^\alpha(h)\star\bar{R}_\alpha(u^i)\star(\partial_i v^j)\star\partial_j \\ &= \mathcal{L}^\star_u(h)\star v + \bar{R}^\alpha(h)\star\nabla^\star_{\bar{R}_\alpha(u)}v\,.\end{aligned} \tag{53}$$

In a next step, we show that the curvature and torsion vanish. The torsion is given by

$$T^\star(\partial_i,\partial_j) = \nabla^\star_i\partial_j - \nabla^\star_j\partial_i - [\partial_i,\partial_j]_\star = 0\,. \tag{54}$$

This is true since the Christoffel symbols are all zero and the derivatives commute. Similarly, we see that the curvature tensor also vanishes:

$$R^\star(\partial_i,\partial_j,\partial_k) = \nabla^\star_i\nabla^\star_j\partial_k - \nabla^\star_{\bar{R}^\alpha(\partial_j)}\nabla^\star_{\bar{R}_\alpha(\partial_i)}\partial_k - \nabla^\star_{[\partial_i,\partial_j]_\star}\partial_k = 0\,. \tag{55}$$

At last, we consider the covariant derivative of the metric:

$$\begin{aligned}\nabla^\star_i g^\star &= \nabla^\star_i(g_{kl}\,dx^k\otimes_\star dx^l) \\ &= \partial_i(g_{kl})dx^k\otimes_\star dx^l - g_{kl}\Gamma^{\star k}_{im}dx^m\otimes_\star dx^l - g_{kl}dx^k\otimes_\star\Gamma^{\star l}_{im}dx^m \\ &= 0\,,\end{aligned} \tag{56}$$

assuming constant metric coefficients and using what we obtain from the star-dual pairing (38), namely:

$$\nabla^\star_i dx^k = -\Gamma^{\star k}_{il}\star dx^l = 0\,.$$

Hence, the choice $\Gamma^{\star k}_{ij} = 0$ is consistent and corresponds to an Einstein space with vanishing curvature.

4.2 $\mathbb{R}^5_q$

The algebra of the 5-dimensional q-deformed plane, $\mathbb{R}^5_q$ is generated by the coordinates $\hat{x}^1,\ldots,\hat{x}^5$ satisfying the relations

$$\begin{aligned}\hat{x}^1\hat{x}^2 &= q\,\hat{x}^2\hat{x}^1,\quad \hat{x}^1\hat{x}^4 = q^{-1}\hat{x}^4\hat{x}^1, \\ \hat{x}^1\hat{x}^5 &= \hat{x}^5\hat{x}^1,\quad \hat{x}^2\hat{x}^4 = \hat{x}^4\hat{x}^2, \\ \hat{x}^2\hat{x}^5 &= q\,\hat{x}^5\hat{x}^2,\quad \hat{x}^4\hat{x}^5 = q^{-1}\hat{x}^5\hat{x}^4.\end{aligned} \tag{57}$$

The coordinate $\hat{x}^3$ is central. Conjugation is given by

$$\hat{x}^{1*} = \hat{x}^5,\ \hat{x}^{2*} = \hat{x}^4,\ \hat{x}^{3*} = \hat{x}^3\,.$$

It follows that the twist (for symmetrical ordering) has the following form

$$\mathcal{F} = \exp\left(\frac{\mathrm{ih}}{2}(\chi_1 \otimes \chi_2 - \chi_2 \otimes \chi_1)\right)\,, \tag{58}$$

where χ_1 and χ_2 are commuting vector fields:

$$\chi_1 = x^2\partial_2 - x^4\partial_4,\ \chi_2 = x^1\partial_1 - x^5\partial_5\,.$$

From the commutation relations, we can see that the metric is given by

$$g^\star = C_{ij}\, d\hat{x}^i \otimes_\star d\hat{x}^j\,, \tag{59}$$

where

$$(C_{ij}) = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

4.2.1 Geometry

Again, we propose

$$\Gamma^{\star k}_{ij} = 0\,, \tag{60}$$

and will show that this definition is a consistent choice. The formula for the covariant derivative (43) reduces to

$$\nabla^\star_z u = \mathcal{L}^\star_z(u^i) \star \hat{\partial}_i\,. \tag{61}$$

This satisfies the axioms for an affine connection since:

- $$\nabla^\star_{u+v} z = \mathcal{L}^\star_{u+v}(z^i) \star \hat{\partial}_i = \mathcal{L}^\star_u(z^i) \star \hat{\partial}_i + \mathcal{L}^\star_v(z^i) \star \hat{\partial}_i = \nabla^\star_u z + \nabla^\star_v z \tag{62}$$

•

$$\nabla^\star_{h\star u} v = \mathcal{L}^\star_{h\star u}(v^i) \star \hat{\partial}_i = h \star \mathcal{L}^\star_u(v^i) \star \hat{\partial}_i = h \star \nabla^\star_u(v) \tag{63}$$

•

$$\begin{aligned} \nabla^\star_u(h \star v) &= \mathcal{L}^\star_u(h \star v^i) \star \hat{\partial}_i \\ &= \mathcal{L}^\star_u(h) \star v + \bar{R}^\alpha(h) \star \mathcal{L}^\star_{\bar{R}_\alpha(u)}(v^i) \star \hat{\partial}_i \\ &= \mathcal{L}^\star(h) \star v + \hat{R}^\alpha(h) \star \nabla^\star_{\bar{R}_\alpha(u)}(v) \end{aligned} \tag{64}$$

The torsion $T^\star$ is given by

$$\begin{aligned} T^\star(u,v) &= \nabla^\star_u v - \nabla^\star_{\bar{R}^\alpha(v)} \bar{R}_\alpha(u) - [u,v]_\star \\ &= \mathcal{L}^\star_u(v^i) \star \hat{\partial}_i - \mathcal{L}^\star_{\bar{R}^\alpha(v)}(\bar{R}_\alpha(u)^i) \star \hat{\partial}_i - [u,v]_\star \,. \end{aligned} \tag{65}$$

Computing the torsion for frame elements, we see explicitly that

$$T^\star(\hat{\partial}_i, \hat{\partial}_j) = 0. \tag{66}$$

This is due to the tensorial property and

$$[\hat{\partial}_i \,, \hat{\partial}_j]_\star = [\bar{f}^\alpha(\hat{\partial}_i), \bar{f}_\alpha(\hat{\partial}_j)] = 0 \,, \tag{67}$$

since the Lie derivative of $\hat{\partial}_i$ along $\bar{f}$, and consequently also along $\bar{R}$, is again a vector field with constant coefficients: $c^j_i \hat{\partial}_j$, $c^j_i \in \mathbb{R}$.
Next, we compute the curvature tensor:

$$\begin{aligned} R^\star(u,v,z) &= \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\bar{R}^\alpha(v)} \nabla^\star_{\bar{R}_\alpha(u)} z - \nabla^\star_{[u,v]_\star} z \\ &= \mathcal{L}^\star_u(\mathcal{L}^\star_v(z^j)) \star \partial_j - \mathcal{L}^\star_{\bar{R}^\alpha(v)}(\mathcal{L}^\star_{\bar{R}_\alpha(u)}(z^j)) \star \partial_j - \mathcal{L}^\star_{[u,v]_\star}(z^j) \star \partial_j \\ &= \mathcal{L}^\star_{u\star v}(z^j) \star \partial_j - \mathcal{L}^\star_{\bar{R}^\alpha(v)\star\bar{R}_\alpha(u)}(z^j) \star \partial_j - \mathcal{L}^\star_{[u,v]_\star}(z^j) \star \partial_j \\ &= \mathcal{L}^\star_{u\star v - \bar{R}^\alpha(v)\star\bar{R}_\alpha(u) - [u,v]_\star}(z^j) \star \partial_j = 0 \,. \end{aligned} \tag{68}$$

The Riemann curvature tensor vanishes identically. In a next step, we show that the chosen connection is a metric one. Let us compute the covariant derivative of the metric:

$$\nabla^\star_i g^\star = \nabla^\star_i (C_{kl} d\hat{x}^k \otimes_\star d\hat{x}^l) \,.$$

From the star-dual pairing (38), we obtain for the covariant derivative of a one-form

$$\nabla_i^\star d\hat{x}^j = 0.$$

Therefore, we are left with

$$\nabla_i^\star g = C_{kl}(\nabla_i^\star d\hat{x}^k \otimes_\star d\hat{x}^l + \bar{R}^\alpha(d\hat{x}^k) \otimes_\star \nabla^\star_{\bar{R}_\alpha(\hat{\partial}_i)} d\hat{x}^l) = 0\,. \tag{69}$$

Metric and connection are compatible.

4.3 $Gl_q(N)$

The quantum space for $Gl_q(N)$ is defined by the relations

$$\hat{x}^i\hat{x}^j = q\hat{x}^j\hat{x}^i,\ i<j \tag{70}$$

for the generators $\hat{x}^i$, $i = 1,\dots N$. In symmetrical ordering, we obtain for the twist

$$\mathcal{F}^{-1} = \exp\left(-\frac{\mathrm{ih}}{2}\sum_{i<j}(\hat{x}^j\hat{\partial}_j \otimes \hat{x}^i\hat{\partial}_i - \hat{x}^i\hat{\partial}_i \otimes \hat{x}^j\hat{\partial}_j)\right)\,. \tag{71}$$

In the same way as before, we can show that the trivial connection satisfies all requirements and defines a connection with vanishing torsion and curvature. There exists no metric invariant under $Gl_q(N)$. Considering the usual Euclidean metric

$$g^\star = \delta_{ij}\ d\hat{x}^i \otimes_\star d\hat{x}^j \tag{72}$$

we find that it is not compatible with the twist, since it is not made of Killing vector fields, and we have

$$\mathcal{L}^\star_{\hat{x}^i\hat{\partial}_i} g^\star \neq 0\,. \tag{73}$$

Nevertheless it constitutes an Einstein space with vanishing curvature.

4.4 Twisted sphere

The twisted sphere [21] is defined by the relations (57) and the additional condition

$$r^2 = 2(\hat{x}^1\hat{x}^5 + \hat{x}^2\hat{x}^4) + (\hat{x}^3)^2\,, \tag{74}$$

with $r \in \mathbb{R}$ being the radius. We will use the stereographic coordinates

$$\phi_1 : S^4_q \;\to\; \mathbb{R}^4_q\,, \tag{75}$$

$$\hat{x}^i \;\mapsto\; y^i = \frac{\hat{x}^i r}{r - \hat{x}^3} = \frac{\hat{x}^i r}{r - \sqrt{r^2 - \kappa_x^2}},\; i = 1,2,4,5\,, \tag{76}$$

with respect to the north pole, where

$$\kappa_x^2 := C_{ij}\hat{x}^i\hat{x}^j, \text{ with } (C_{ij}) = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}. \tag{77}$$

The inverse transformation is given by

$$\phi_1^{-1} : y^i \;\mapsto\; \hat{x}^i = \frac{2r^2 y^i}{r^2 + \kappa_y^2}\,, \tag{78}$$

$$\hat{x}^3 = r\frac{\kappa_y^2 - r^2}{r^2 + \kappa_y^2}\,. \tag{79}$$

Both, r^2 and κ_x^2 are central in the coordinate algebra. The induced metric in these coordinates is given by

$$g^\star = \frac{4r^4}{(r^2 + \kappa_y^2)^2} \star C_{ij} dy^i \otimes_\star dy^j\,, \tag{80}$$

where $\kappa_y^2 = C_{ij}\, y^i y^j$. Furthermore, the coordinates y^i satisfy the same commutation relations as $\hat{x}^i$ given in (57). Therefore, also the twist is unchanged.

In order to simplify the notation, we introduce the following definitions: For the vector fields, let us define

$$t_i := y^i \frac{\partial}{\partial y^i} \equiv y^i \partial_i, \tag{81}$$

note that here no summation over the index i is implied. Hence, we can write for the twist (58)

$$\mathcal{F} = \exp\left(-\frac{\mathrm{ih}}{2}\sum_{ij}\varphi_{ij}\, t_i \otimes t_j\right), \tag{82}$$

with

$$\varphi_{ij} = -\varphi_{ji} = -\varphi_{ij'}, \tag{83}$$

$$\varphi_{12} = 1,\ \varphi_{ii} = \varphi_{ii'} = \varphi_{3i} = 0, \tag{84}$$

and $i' := 6 - i$. Furthermore, let us introduce P_{ij} and its square,

$$P_{ij} := e^{\frac{\mathrm{ih}}{2}\varphi_{ij}},\ q_{ij} := P_{ij}^2, \tag{85}$$

where in the latter definition no matrix multiplication is implied, i.e., $q_{ij} = P_{ij}P_{ij}$, again with no index summation. Using these definitions, the metric (80) reads

$$g^* = \sum_{ij} g^*_{ij} dy^i \otimes_* dy^j = \frac{4r^2}{(r^2+\kappa_y^2)^2}\sum_{i,j} C_{ij}P_{ij}\, dy^i \otimes dy^j . \tag{86}$$

In the second step, we have used the usual definition for Lie derivatives acting on 1-forms:

$$t_i(dy^k) = \delta_{ik} dy^i ,$$

no summation over the indices is implied. The Levi-Civita connection can be obtained by demanding vanishing torsion and vanishing covariant derivative of the metric. The former condition reads

$$\Gamma^{*\,k}_{ij} = q_{ij}\Gamma^{*\,k}_{ji} . \tag{87}$$

The latter condition then leads to

$$\Gamma^{*\,k}_{ij} = \frac{1}{2}\left(q_{ij}\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \star g^{*lk} . \tag{88}$$

As a result, the universal connection is the same as in the undeformed case:

$$\nabla^* = \nabla . \tag{89}$$

The converse is also true: Assuming (89), we obtain (88) for the connection coefficients.

Similarily, we obtain for the Riemann curvature:

$$R^* = R\,, \tag{90}$$

and in terms of components

$$\begin{aligned} R^* &= R^*_{ikl}{}^m dy^i \otimes_* dy^j \otimes_* dy^k \otimes_* \partial_m\,, & (91)\\ R^*_{ijkl} &= \frac{1}{r^2}\left(g_{li}g_{jk} - q_{ik}\, g_{lj}g_{ik}\right)\,. & (92) \end{aligned}$$

We have considered a stereographic projection of the twisted sphere from the north pole. Of course, equally well we could have chosen a projection from the south pole:

$$\begin{aligned} \phi_2 : S^4_q &\to \mathbb{R}^4_q\,, & (93)\\ \hat{x}^i &\mapsto \frac{\hat{x}^i r}{r+\hat{x}^3}\,. \end{aligned}$$

This formula enables us to construct the map from one projection to the other:

$$\phi_2 \circ \phi_1^{-1} : y^i \mapsto \frac{r^2}{\kappa_y^2} y^i\,. \tag{94}$$

One can show that this map leaves the metric (80) invariant. So can freely wander from one projection to the other.

4.5 Conformal transformations and constraints on NC corrections

In this Section, we want to discuss some possibilities of constructing a non-trivial, noncommutative metric describing an Einstein space.

Let $(\mathcal{M}, g)$ be a pseudo-Riemannian 2 dimensional manifold with metric g and f some function on $\mathcal{M}$. After a conformal transformation of the metric,

$$\tilde{g} = \mathrm{e}^{2\mathrm{f}}\mathrm{g}\,, \tag{95}$$

the components of the Ricci tensor corresponding to the metric $\tilde{g}$ can be expressed as [22]

$$\widetilde{R}_{ij} = R_{ij} - (\partial_k \partial^k f)\, g_{ij}\,, \tag{96}$$

where R_{ij} corresponds to the metric g.

Let us generalize (96) to noncommutative spaces. For simplicity, let us consider Weyl-Moyal deformation and let us assume a noncommutative metric with constant components g_{ij}. This means that the corresponding Riemann tensor vanishes. As before, we take a conformal transformation

$$\tilde{g}_{kl} = \mathrm{e}_\star^{2\mathrm{f}} \mathrm{g}_{\mathrm{kl}}\,. \tag{97}$$

For simplicity of notation, we drop the star on the noncommutative metric and on the objects derived from it in this section. The star-exponential function $\mathrm{e}_\star^{\mathrm{g}}$ is defined by

$$\mathrm{e}_\star^{\mathrm{g}} = 1 + \mathrm{g} + \frac{1}{2}\,\mathrm{g}\star\mathrm{g} + \frac{1}{6}\,\mathrm{g}\star\mathrm{g}\star\mathrm{g} + \ldots$$

Explicit calculation of the Ricci tensor yields:

$$\tilde{R}_{ij} = -\frac{1}{2} g_{ij}\partial_k(e_\star^{-2f}\star(\partial^k e_\star^{2f})) + \frac{1}{2}\partial_i(e_\star^{-2f}\star(\partial_j e_\star^{2f})) - \frac{1}{2}\partial_j(e_\star^{-2f}\star(\partial_i e_\star^{2f}))\,. \tag{98}$$

Starting from a space with vanishing curvature, we would like to construct a metric which defines an Einstein space with a nontrivial Ricci tensor. An Einstein space is defined by the relation

$$\widetilde{Ric} = c\tilde{g}\,,$$

where c is a constant. This means that the sum of the last two terms of (98) have to vanish (and the coefficient of the first term needs to be constant). Therefore, we obtain the following condition on the "conformal weight" f:

$$\partial_i e_\star^{-2f} \star \partial_j e_\star^{2f} = \partial_j e_\star^{-2f} \star \partial_i e_\star^{2f}\,. \tag{99}$$

This condition can be simplified by the use of operators D connecting different star products or also star products to the pointwise product[1], as

[1] D is a formal power series in the noncommutativity parameters of differential operators.

discussed e.g. in [19]:

$$D(h \star f) = (Dh) \cdot (Df) \,. \tag{100}$$

The defining Eq. (100), leads to the following expression for the star inverse of the function h:

$$h_\star^{-1} = D^{-1}(Dh)^{-1} \,, \qquad \text{and} \qquad De_\star^g = e^{Dg} \,. \tag{101}$$

Now, we can simplify Eq. (99) and obtain

$$\partial_i f \star \partial_j f = \partial_j f \star \partial_i f \,. \tag{102}$$

The solutions of this equation classify all noncommutative Einstein spaces obtained by a conformal transformation from a flat geometry. In two commutative dimensions, the only Einstein spaces are the flat space, the sphere and the hyperboloid.

A different, but straight forward approach to obtain non-trivial noncommutative Einstein spaces is to make a perturbative ansatz. We start from the round metric to zeroth order in θ, in stereographic coordinates:

$$g_{ij} = \frac{4}{(1+x^2+y^2)^2}\delta_{ij} + g_{ij}^{(1)} + g_{ij}^{(2)} + \mathcal{O}(\theta^3) \,. \tag{103}$$

Higher order corrections (in θ) denoted by $g^{(1)}$, $g^{(2)}$, ... are constrained by the wish to obtain an Einstein space. The constraints are obtained by direct computation of the Riemann and Ricci tensor, respectively. We could also start from a hyperbolic or flat metric and consider higher order corrections. Even a pure quantum metric would be conceivable, which vanishes in the limit $\theta \to 0$.

For convenience, we define

$$g_{ij}^{(0)} \equiv \frac{4}{(1+x^2+y^2)^2}\delta_{ij} \,. \tag{104}$$

In a first step, let us compute the star-inverse metric, up to second order. It satisfies the condition

$$g_{ij} \star g^{jk} = \delta_i^k \,. \tag{105}$$

We expand the above condition to second order and use the following ansatz for the inverse metric:

$$g^{ij} = h^{(0)ij} + h^{(1)ij} + h^{(2)ij} + \mathcal{O}(\theta^3)\,. \tag{106}$$

To zeroth order, we obviously find

$$h^{(0)ij} = \frac{(1+x^2+y^2)^2}{4}\delta^{ij}\,. \tag{107}$$

For the first order term of the inverse metric we get

$$h^{(1)ij} = -\frac{(1+x^2+y^2)^4}{16}g^{(1)ij}\,, \tag{108}$$

whereas the second order reads:

$$\begin{aligned} h^{(2)ij} = &-\frac{(1+x^2+y^2)^4}{16}g^{(2)ij} + \frac{(1+x^2+y^2)^6}{64}g_l^{(1)i}g^{(1)lj} \\ &- \delta^{ij}\theta^2\frac{1-x^2-y^2}{1+x^2+y^2}\,. \end{aligned} \tag{109}$$

We have assumed the following form of the noncommutativity parameters:

$$\theta^{ab} = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \tag{110}$$

and that the components of the metric depend on the radius r^2 only, i.e.,

$$g_{ij}^{(1)} = g_{ij}^{(1)}(r^2)\,. \tag{111}$$

The indices are pulled up and down by the Kronecker delta. Summing up, we have

$$\begin{aligned} g^{ij} = &\frac{(1+x^2+y^2)^4}{16}\left(g^{(0)ij} - g^{(1)ij} - g^{(2)ij}\right) \\ &+ \frac{(1+x^2+y^2)^6}{64}g_l^{(1)i}g^{(1)lj} - \delta^{ij}\theta^2\frac{1-x^2-y^2}{1+x^2+y^2} + \mathcal{O}(\theta^3)\,. \end{aligned} \tag{112}$$

We want to compute the Ricci tensor and obtain restrictions on the corrections for the metric. Therefore, we need to compute the Christoffel symbols first. They are given by

$$\Gamma_{ij}^{k} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \star g^{lk}\,. \tag{113}$$

Similarly as before, we expand the Christoffel symbols up to second order,

$$\Gamma_{ij}^{k} = \Gamma_{ij}^{(0)k} + \Gamma_{ij}^{(1)k} + \Gamma_{ij}^{(2)k} + \mathcal{O}(\theta^3)\,, \tag{114}$$

where $\Gamma_{ij}^{(0)k}$ are the well-known commutative components. For the first order symbols one obtains

$$\begin{aligned}\Gamma_{ij}^{(1)k} = {} & \frac{\mathrm{i}}{4}\theta^{mn}\left(\partial_m\partial_i g_{jl}^{(0)} + \partial_m\partial_j g_{il}^{(0)} - \partial_m\partial_l g_{ij}^{(0)}\right)\partial_n h^{(0)lk} \\ & + \frac{1}{2}\left(\partial_i g_{jl}^{(0)} + \partial_j g_{il}^{(0)} - \partial_l g_{ij}^{(0)}\right) h^{(1)lk} \\ & + \frac{1}{2}\left(\partial_i g_{jl}^{(1)} + \partial_j g_{il}^{(1)} - \partial_l g_{ij}^{(1)}\right) h^{(0)lk}\,,\end{aligned} \tag{115}$$

the second order is given by

$$\begin{aligned}\Gamma_{ij}^{(2)k} = {} & -\frac{1}{8}\theta^{m_1n_1}\theta^{m_2n_2}\left(\partial_{m_1}\partial_{m_2}\partial_i g_{jl}^{(0)} + \partial_{m_1}\partial_{m_2}\partial_j g_{il}^{(0)} - \partial_{m_1}\partial_{m_2}\partial_l g_{ij}^{(0)}\right)\partial_{n_1}\partial_{n_2} h^{(0)lk} \\ & + \frac{\mathrm{i}}{4}\theta^{mn}\left(\partial_m\partial_i g_{jl}^{(0)} + \partial_m\partial_j g_{il}^{(0)} - \partial_m\partial_l g_{ij}^{(0)}\right)\partial_n h^{(1)lk} \\ & + \frac{\mathrm{i}}{4}\theta^{mn}\left(\partial_m\partial_i g_{jl}^{(1)} + \partial_m\partial_j g_{il}^{(1)} - \partial_m\partial_l g_{ij}^{(1)}\right)\partial_n h^{(0)lk} \\ & + \frac{1}{2}\left(\partial_i g_{jl}^{(2)} + \partial_j g_{il}^{(2)} - \partial_l g_{ij}^{(2)}\right) h^{(0)lk} + \frac{1}{2}\left(\partial_i g_{jl}^{(1)} + \partial_j g_{il}^{(1)} - \partial_l g_{ij}^{(1)}\right) h^{(1)lk} \\ & + \frac{1}{2}\left(\partial_i g_{jl}^{(0)} + \partial_j g_{il}^{(0)} - \partial_l g_{ij}^{(0)}\right) h^{(2)lk}\,.\end{aligned} \tag{116}$$

In the next step, we compute the Riemann tensor defined by

$$R_{ijk}{}^{l} = \partial_i\Gamma_{jk}^{l} - \partial_j\Gamma_{ik}^{l} + \Gamma_{jk}^{m} \star \Gamma_{im}^{l} - \Gamma_{ik}^{m} \star \Gamma_{jm}^{l}\,. \tag{117}$$

Expanding the Riemann tensor

$$R_{ijk}{}^{l} = R_{ijk}^{(0)l} + R_{ijk}^{(1)l} + R_{ijk}^{(2)l} + \mathcal{O}(\theta^3)\,, \tag{118}$$

we obtain to first order

$$R^{(1)\,l}_{ijk} = \partial_i \Gamma^{(1)l}_{jk} - \partial_j \Gamma^{(1)l}_{ik} + \Gamma^{(1)m}_{jk}\Gamma^{(0)l}_{im} + \Gamma^{(0)m}_{jk}\Gamma^{(1)l}_{im} - \Gamma^{(1)m}_{ik}\Gamma^{(0)l}_{jm} - \Gamma^{(0)m}_{ik}\Gamma^{(1)l}_{jm} + \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)s}_{jk}\partial_n \Gamma^{(0)l}_{is} - \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)s}_{ik}\partial_n \Gamma^{(0)l}_{js} \,. \tag{119}$$

The second order of the Riemann tensor is given by

$$\begin{aligned} R^{(2)\,l}_{ijk} =& \partial_i \Gamma^{(2)l}_{jk} - \partial_j \Gamma^{(2)l}_{ik} + \Gamma^{(2)m}_{jk}\Gamma^{(0)l}_{im} + \Gamma^{(1)m}_{jk}\Gamma^{(1)l}_{im} + \Gamma^{(0)m}_{jk}\Gamma^{(2)l}_{im} \\ & - \Gamma^{(2)m}_{ik}\Gamma^{(0)l}_{jm} - \Gamma^{(1)m}_{ik}\Gamma^{(1)l}_{jm} - \Gamma^{(0)m}_{ik}\Gamma^{(2)l}_{jm} \\ & + \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(1)s}_{jk}\partial_n \Gamma^{(0)l}_{is} + \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)s}_{jk}\partial_n \Gamma^{(1)l}_{is} \\ & - \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(1)s}_{ik}\partial_n \Gamma^{(0)l}_{js} - \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)s}_{ik}\partial_n \Gamma^{(1)l}_{js} \\ & - \frac{1}{8}\theta^{m_1 n_1}\theta^{m_2 n_2}\partial_{m_1}\partial_{m_2}\Gamma^{(0)m}_{jk}\partial_{n_1}\partial_{n_2}\Gamma^{(0)l}_{im} \\ & + \frac{1}{8}\theta^{m_1 n_1}\theta^{m_2 n_2}\partial_{m_1}\partial_{m_2}\Gamma^{(0)m}_{ik}\partial_{n_1}\partial_{n_2}\Gamma^{(0)l}_{jm} \,. \end{aligned} \tag{120}$$

To zeroth order, we of course obtain the undeformed Riemann tensor. The Ricci tensor is obtained by a suitable contraction of indices:

$$R_{ij} = R_{kij}{}^{k} \,. \tag{121}$$

For a noncommutative Einstein space, Ricci tensor and metric have to be proportional:

$$R^{(i)}_{mn} = c g^{(i)}_{mn} \,, \tag{122}$$

for all i, with some constant c. From the well-known zeroth order result

$$R^{(0)}_{ij} = g^{(0)}_{ij} \,, \tag{123}$$

it follows that $c = 1$. The constraining conditions on the corrections $g^{(1)}_{ij}$ and $g^{(2)}_{ij}$ are very lengthy. Therefore, we will only provide the first order

expressions explicitly. It is given by

$$
\begin{aligned}
R^{(1)}_{ij} = R^{(1)\,k}_{kij} &= \frac{1}{2}\left(\partial_k\partial_j g^{(0)}_{il} - \partial_k\partial_l g^{(0)}_{ij} - \partial_i\partial_j g^{(0)}_{kl} + \partial_i\partial_l g^{(0)}_{kj}\right) h^{(1)lk} \\
&+ \frac{1}{2}\left(\partial_j g^{(0)}_{il} - \partial_l g^{(0)}_{ij}\right)\partial_k h^{(1)lk} - \frac{1}{2}\left(\partial_j g^{(0)}_{kl} - \partial_l g^{(0)}_{kj}\right)\partial_i h^{(1)lk} \\
&+ \frac{1}{2}\left(\partial_k\partial_j g^{(1)}_{il} - \partial_k\partial_l g^{(1)}_{ij} - \partial_i\partial_j g^{(1)}_{kl} + \partial_i\partial_l g^{(1)}_{kj}\right) h^{(0)lk} \\
&+ \frac{1}{2}\left(\partial_j g^{(1)}_{il} - \partial_l g^{(1)}_{ij}\right)\partial_k h^{(0)lk} - \frac{1}{2}\left(\partial_j g^{(1)}_{kl} - \partial_l g^{(1)}_{kj}\right)\partial_i h^{(0)lk} \\
&+ \frac{\mathrm{i}}{4}\theta^{mn}\left(\partial_k\partial_m\partial_j g^{(0)}_{il} - \partial_k\partial_m\partial_l g^{(0)}_{ij} - \partial_i\partial_m\partial_j g^{(0)}_{kl} + \partial_i\partial_m\partial_l g^{(0)}_{kj}\right)\partial_n h^{(0)lk} \\
&+ \frac{\mathrm{i}}{4}\theta^{mn}\left(2\partial_m\partial_i g^{(0)}_{jl} + \partial_m\partial_j g^{(0)}_{il} - \partial_m\partial_l g^{(0)}_{ij}\right)\partial_k\partial_n h^{(0)lk} \\
&- \frac{\mathrm{i}}{4}\theta^{mn}\left(\partial_m\partial_j g^{(0)}_{kl} - \partial_m\partial_l g^{(0)}_{kj}\right)\partial_i\partial_n h^{(0)lk} \\
&+ \Gamma^{(0)k}_{km}\Bigg(\frac{\mathrm{i}}{4}\theta^{ab}\left(\partial_a\partial_i g^{(0)}_{jl} + \partial_a\partial_j g^{(0)}_{il} - \partial_a\partial_l g^{(0)}_{ij}\right)\partial_b h^{(0)lm} \\
&+ \frac{1}{2}\left(\partial_i g^{(0)}_{jl} + \partial_j g^{(0)}_{il} - \partial_l g^{(0)}_{ij}\right) h^{(1)lm} + \frac{1}{2}\left(\partial_i g^{(1)}_{jl} + \partial_j g^{(1)}_{il} - \partial_l g^{(1)}_{ij}\right) h^{(0)lm}\Bigg) \\
&+ \Gamma^{(0)m}_{ij}\Bigg(\frac{\mathrm{i}}{4}\theta^{ab}\left(\partial_a\partial_k g^{(0)}_{ml} + \partial_a\partial_m g^{(0)}_{kl} - \partial_a\partial_l g^{(0)}_{km}\right)\partial_b h^{(0)lk} \\
&+ \frac{1}{2}\left(\partial_k g^{(0)}_{ml} + \partial_m g^{(0)}_{kl} - \partial_l g^{(0)}_{km}\right) h^{(1)lk} + \frac{1}{2}\left(\partial_k g^{(1)}_{ml} + \partial_m g^{(1)}_{kl} - \partial_l g^{(1)}_{km}\right) h^{(0)lk}\Bigg) \\
&- \Gamma^{(0)k}_{im}\Bigg(\frac{\mathrm{i}}{4}\theta^{ab}\left(\partial_a\partial_k g^{(0)}_{jl} + \partial_a\partial_j g^{(0)}_{kl} - \partial_a\partial_l g^{(0)}_{kj}\right)\partial_b h^{(0)lm} \\
&+ \frac{1}{2}\left(\partial_k g^{(0)}_{jl} + \partial_j g^{(0)}_{kl} - \partial_l g^{(0)}_{kj}\right) h^{(1)lm} + \frac{1}{2}\left(\partial_k g^{(1)}_{jl} + \partial_j g^{(1)}_{kl} - \partial_l g^{(1)}_{kj}\right) h^{(0)lm}\Bigg) \\
&- \Gamma^{(0)m}_{kj}\Bigg(\frac{\mathrm{i}}{4}\theta^{ab}\left(\partial_a\partial_i g^{(0)}_{ml} + \partial_a\partial_m g^{(0)}_{il} - \partial_a\partial_l g^{(0)}_{im}\right)\partial_b h^{(0)lk} \\
&+ \frac{1}{2}\left(\partial_i g^{(0)}_{ml} + \partial_m g^{(0)}_{il} - \partial_l g^{(0)}_{im}\right) h^{(1)lk} + \frac{1}{2}\left(\partial_i g^{(1)}_{ml} + \partial_m g^{(1)}_{il} - \partial_l g^{(1)}_{im}\right) h^{(0)lk}\Bigg) \\
&+ \frac{\mathrm{i}}{2}\theta^{ab}\partial_a\Gamma^{(0)m}_{ij}\partial_b\Gamma^{(0)k}_{km} - \frac{\mathrm{i}}{2}\theta^{ab}\partial_a\Gamma^{(0)m}_{kj}\partial_b\Gamma^{(0)k}_{im} \equiv g^{(1)}_{ij}\,.
\end{aligned}
$$

For the sake of simplicity of presentation, we provide the constraint for the second order contribution in terms of the noncommutative corrections to the Christoffel symbols (114) only:

$$
\begin{aligned}
R^{(2)\,k}_{ijk} = R^{(2)}_{ij} = {} & \partial_k \Gamma^{(2)k}_{ij} - \partial_i \Gamma^{(2)k}_{kj} + \Gamma^{(2)m}_{ij}\Gamma^{(0)k}_{km} + \Gamma^{(1)m}_{ij}\Gamma^{(1)k}_{km} + \Gamma^{(0)m}_{ij}\Gamma^{(2)k}_{km} \\
& - \Gamma^{(2)m}_{kj}\Gamma^{(0)k}_{im} - \Gamma^{(1)m}_{kj}\Gamma^{(1)k}_{im} - \Gamma^{(0)m}_{kj}\Gamma^{(2)k}_{im} \\
& + \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(1)m}_{ij}\partial_n \Gamma^{(0)k}_{km} + \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)m}_{ij}\partial_n \Gamma^{(1)k}_{km} \\
& - \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(1)m}_{kj}\partial_n \Gamma^{(0)k}_{im} - \frac{\mathrm{i}}{2}\theta^{mn}\partial_m \Gamma^{(0)m}_{kj}\partial_n \Gamma^{(1)k}_{im} \\
& - \frac{1}{8}\theta^{m_1 n_1}\theta^{m_2 n_2}\partial_{m_1}\partial_{m_2}\Gamma^{(0)m}_{ij}\partial_{n_1}\partial_{n_2}\Gamma^{(0)k}_{km} \\
& + \frac{1}{8}\theta^{m_1 n_1}\theta^{m_2 n_2}\partial_{m_1}\partial_{m_2}\Gamma^{(0)m}_{kj}\partial_{n_1}\partial_{n_2}\Gamma^{(0)k}_{im} \equiv g^{(2)}_{ij} \,.
\end{aligned}
\tag{124}
$$

5 Noncommutative astrophysics

Let us turn to applications of noncommutative ideas to astrophysics and cosmology. If noncommutativity exists in nature, it seems that its effects can be observed more easily in astrophysical scenarios than in collider experiments. The former always include huge distances and small effects can add up. In this and the next section, I want to present a biased selection of examples on noncommutative astrophysics and cosmology. We start with a discussion of an exact solution of the noncommutative Einstein equations. These equations are fairly complicated, since they involve the star product. Therefore, only very few explicit solutions are known.

In the simplest case, the twist consists of Killing vectors, i.e., of vector fields K_α with $\mathcal{L}_{K_\alpha} g^\star = 0$. Let us denote the Lie algebra of (affine) Killing vector fields by $\mathbf{g}_K$, then the twist is an element of the tensor product of the respective universal senveloping algebras,

$$
\mathcal{F} \in \mathcal{U}\mathbf{g}_K \otimes \mathcal{U}\mathbf{g}_K \,,
$$

then the deformed Einstein equations reduce to the undeformed ones [23].

Some explicit examples are studied in [12, 20, 24], where we have showed that deformed Einstein spaces (the Weyl-Moyal plane and the twisted sphere) are noncommutative Einstein spaces.

Schupp and Solodukhin [25] constructed a spherical symmetric vacuum solution (black hole) in (3+1)D using the fuzzy sphere as the underlying noncommutative space:

$$[\hat{x}^a, \hat{x}^b] = \mathrm{i}\lambda(\hat{\rho})\epsilon_{abc}\hat{x}^c\,, \tag{125}$$

where $a, b, c = 1, 2, 3$, and λ denotes a function of the Casimir operator $\hat{\rho} = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$ in the coordinate algebra. The corresponding twist is given by:

$$\mathcal{F} = \sum_n C_n(\frac{\lambda}{\rho})\mathcal{L}^n_{\xi_+} \otimes \mathcal{L}^n_{\xi_-}\,, \tag{126}$$

where $\xi_\pm$ are rotation generators. Since the rotation generators are Killing vectors, the solution is the classical one except for the parameters (eigenvalues of $\hat{\rho}$) $\rho = n\lambda$, $n = 2j = 0, 1, 2, \ldots$:

$$g_{00} = \frac{a}{r} - 1\,,\; g_{0,i} = 0\,,\; g_{ij} = \frac{r^2}{\rho^2}\delta_{ij}\,, \tag{127}$$

where $r = \frac{1}{\rho}(\rho + \frac{a}{4})$. Some remarks are in order:

- The above solution is valued outside the horizon $r = a$ or $\rho = a/4$. Therefore, $n > a/(4\,\lambda)$.
- The noncommutative space-time is sliced by fuzzy spheres with radius ρ.
- Inside the horizon, they have found a different solution:

$$g_{00} = 1 - \frac{a}{r}\,,\; g_{0,i} = 0\,,\; g_{ij} = \frac{r^2}{\rho^2}(\delta_{ij} - \frac{x_i x_j}{\rho^2})\,, \tag{128}$$

  where $\rho^2 = \frac{a^2}{2r}\frac{1}{a-r}$.

  The classical singularity, $r = 0$, corresponds to $\rho = \infty$, therefore the singularity does not disappear, but it is replaced by a sequence of fuzzy spheres. One hope of noncommutative geometry is that singularities are dissolved. However, in this case this is not true.

Noncommutative theories in general lead to modified dispersion relations of the kind

$$E^2 = p^2c^2(1 + \lambda E)^2 + m^2c^4\,, \tag{129}$$

where λ is the noncommutative deformation parameter. Bertolami and Zarro [26] used the modified dispersion relation (129) in order to calculate corrections to energy and particle density and to the pressure of a photon gas, a non-relativistic ideal gas and a degenerate fermion gas. These results are applied to study the stability of stars. There, the gas and radiation pressure

$$P = P_{rad} + P_{gas} \tag{130}$$

are in equilibrium with gravitational force. Let us choose for $\lambda = 2.5 \cdot 10^{-19} GeV^{-1}$. Then, they obtain the following results for the ratio $P_{\mathrm{NC}}/P_{\mathrm{C}}$ (the index distinguishes between the noncommutative and the commutative case):

- Main sequence stars: $\frac{P_{\mathrm{NC}}}{P_{\mathrm{C}}} = 10^{-25}$.
- White dwarfs: $\frac{P_{\mathrm{NC}}}{P_{\mathrm{C}}} = 10^{-22}$.
- Neutron stars: $\frac{P_{\mathrm{NC}}}{P_{\mathrm{C}}} = 10^{-19}$.

This means that the stability regions of stars are enlarged. The effect increases with the mass of the stars. This calculation may not be very realistic, since it relies on many approximations and assumptions. But points into the right direction and gives some hints what the effects of noncommutativity could be.

6 NC cosmology

Another example provide γ-ray bursts in active galaxies, see e.g. Amelino-Camelia and Majid [27]. Due to modified dispersion relations (129), there is a difference in the arrival time of photons with different energies:

$$|\delta t| \approx \lambda \frac{L}{c} \delta\omega\,,$$

where L denotes the distance of the galaxy, $\delta\omega$ the energy range of the burst. So far, an effect like this has not been detected. This leads to bounds for noncommutativity parameter λ.

Ohl, Schenkel [28] discuss more general twists of the form

$$\mathcal{F} = \exp(-\frac{ih}{2}\theta^{ab} X_a \otimes X_b)\,, \tag{131}$$

$\theta^{ab} = const$ and X_a, X_b any vector fields satisfying

$$[X_a, X_b] = 0\,. \tag{132}$$

Starting from symmetry arguments, they classified the possible twists leading to deformed black hole models (spherical symmetric) and to deformed cosmological (Friedman-Robertson-Walker) models.

A lot of efforts have also been spent on the very interesting topic of noncommutative inflation. Unfortunately, this will not be covered here, for a reference see e.g. [29–31].

7 Summary

In my opinion, noncommutative ideas can best be tested in astrophysical scenarios. Up to now, there are some predictions, but contact to "real" physics and experiments is not yet established. The presented model of twisted gravity is an interesting candidate implementing noncommutativity into gravitational physics. However, matter has not been included so far and resembles a further step in the development. This might allow the study of effects like cosmic microwave background or Hawking radiation within this framework.

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